COMP1215 Foundations of Computer Science

(A short introduction to) combinatorics & combinatorial analysis

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Content

Pre-course information

- What is combinatorics: the fancy name of counting
- Warning: counting is hard

  $\because$ it is one of the most difficult area in mathematics
  $\because$ it is universal $\implies$ important for computer science
- Study material: slides + $\underbrace{\text{madbook + reading + watch video}}_{\text{self learning}}$
- Book
  - *Discrete Mathematics and Its Applications*
    by Kenneth Rosen, enough for this course
  - *Concrete mathematics: a foundation for computer science*
    by Graham, Knuth & Patashnik, classic
  - *Counting: The art of enumerative combinatorics*
    by Martin, nice
  - *Enumerative Combinatorics Volume 1*
    by Richard Stanley, advanced
  - Combinatorics Through Guided Discovery
    by Bogart, free book
  - *Schaum's Outline of Combinatorics*
    for more practise problems
  - *102 Combinatorial Problems: From the Training of the USA IMO Team*
  - A stackexhcange post on advanced books
    if plan to do a PhD
- Outcome: become *less ignorant* in counting

Prerequisite: Set theory (I assume you know)

- $X := \{x \mid \text{set description}\}$ — set
- universal set U and empty set $\varnothing$ — special sets
- $x \in X, x \notin X$ — membership
- $X^c := \{x \mid x \notin X\}$ — complement
- $|X| := \#\{x \mid x \in X\}$ — cardinality
- X is finite if $|X| < +\infty$ — finite set
- $X \cup Y := \{x \mid x \in X \text{ OR } x \in Y\}$ — union
- $X \cap Y := \{x \mid x \in X \text{ AND } x \in Y\}$ — intersection
- $X \setminus Y := \{x \mid x \in X \text{ AND } x \notin Y\}$ — complement
- $X \times Y := \{(x, y) \mid x \in X \text{ AND } y \in Y\}$ — Cartesian product
- X, Y disjoint iff $X \cap Y = \varnothing$
  - $|X \cup Y| = |X| + |Y|$
- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ — natural number, integer, reals
- Function: injection, surjection, bijection

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Inclusion-exclusion principle

$\boxed{|X \cup Y| = |X| + |Y| \iff X, Y \text{ disjoint}}$

- For two sets: $|A \cup B| \;=\; |A| + |B| - |A \cap B|$

  Proof

$$\begin{aligned} |A \cup B| &= |A \cup (B \setminus A)| \overset{\text{disjoint}}{=} |A| + |B \setminus A| && (*) \\ |B| &= |(B \setminus A) \cup (B \cap A)| \overset{\text{disjoint}}{=} |B \setminus A| + |B \cap A| && (\dagger) \end{aligned}$$

  $(*) - (\dagger)$ gives .

- For three sets $|A \cup B \cup C| \;=\; |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

  Proof: recursion (use the previous case)

$$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)| = |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$

  By the distributive property $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - \Big[|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \Big] \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \end{aligned}$$

  where $|A \cap B \cap A \cap C| \overset{\text{commutative}}{=} |A \cap A \cap B \cap C| \overset{\text{idempotent}}{=} |A \cap B \cap C|$.

Inclusion-exclusion principle of four sets

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \tag{2}$$

- **Proof**: recursion (use the previous cases)

$$\begin{aligned}
|A \cup B \cup C \cup D| &= |A \cup (B \cup C \cup D)| \\
&\overset{(1)}{=} |A| + |B \cup C \cup D| - |A \cap (B \cup C \cup D)| \\
&\overset{(2)}{=} |A| + |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D| - |A \cap (B \cup C \cup D)|
\end{aligned}$$

For the last term

$$\begin{aligned}
|A \cap (B \cup C \cup D)| &\overset{\text{distributive}}{=} |(A \cap B) \cup (A \cap C) \cup (A \cap D)| \\
&\overset{(2)}{=} \begin{array}{l} +|A \cap B| - |(A \cap B) \cap (A \cap C)| \\ +|A \cap C| - |(A \cap B) \cap (A \cap D)| + |(A \cap B) \cap (A \cap C) \cap (A \cap D)| \\ +|A \cap D| - |(A \cap C) \cap (A \cap D)| \end{array}
\end{aligned}$$

Therefore

$$|A \cup B \cup C \cup D| = \begin{array}{llll} +|A| & -|A \cap B| & +|A \cap B \cap C| & -|A \cap B \cap C \cap D| \\ +|B| & -|A \cap C| & +|A \cap B \cap D| & \\ +|C| & -|A \cap D| & +|A \cap C \cap D| & \\ +|D| & -|B \cap C| & +|B \cap C \cap D| & \\ & -|B \cap D| & & \\ & -|C \cap D| & & \end{array}$$

- Key idea here: **recursion** (use the previous cases to solve the current case)

A pattern

- 0 set $|\varnothing| = 0$
- 1 set $|A| = |A|$.
- 2 sets $|A \cup B| \; = \; |A| + |B| - |A \cap B|$.
- 3 sets $|A \cup B \cup C| \; = \; |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
- 4 sets

$$
\begin{array}{rcllll}
|A \cup B \cup C \cup D| & = & +|A| & -|A \cap B| & +|A \cap B \cap C| & -|A \cap B \cap C \cap D| \\
& & +|B| & -|A \cap C| & +|A \cap B \cap D| & \\
& & +|C| & -|A \cap D| & +|A \cap C \cap D| & \\
& & +|D| & -|B \cap C| & +|B \cap C \cap D| & \\
& & & -|B \cap D| & & \\
& & & -|C \cap D| & &
\end{array}
$$

	single term	double term	triple term	quadruple term
$\lvert A\rvert$	1			
$\lvert A \cup B\rvert$	2	1		
$\lvert A \cup B \cup C\rvert$	3	3	1	
$\lvert A \cup B \cup C \cup D\rvert$	4	6	4	1

Later we will see that these are binomial coefficients.

Multiplying polynomial

- Consider $(1+x)^2$

$$(1+x)^2 \;=\; 1x^0 + 2x^1 + 1x^2$$

coefficients $\{1, 2, 1\}$

- Consider $(1+x)^3$

$$(1+x)^3 \;=\; 1x^0 + 3x^1 + 3x^2 + 1x^3$$

coefficients $\{1, 3, 3, 1\}$

- Consider $(1+x)^4$

$$(1+x)^4 \;=\; 1x^0 + 4x^1 + 6x^2 + 4x^3 + 1x^4$$

coefficients $\{1, 4, 6, 4, 1\}$

	power-1 term	power-2 term	power-3 term	power-4 term
$(1+x)^1$	1			
$(1+x)^2$	2	1		
$(1+x)^3$	3	3	1	
$(1+x)^4$	4	6	4	1

we can count things using the coefficients of polynomials ($\because$ they have a bijection)

What's the big deal of inclusion-exclusion principle? Not in exam

- **Theorem (Generalized inclusion-exclusion principle)** For any finite sequence $A_1, \ldots, A_n$ of $n \geq 2$ subsets of a finite set X, we have
$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \varnothing}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|.$$
- Application 1: The number of surjections, S_{np}, between a n-set A and a p-set B, where $n \geq p$, is
$$S_{np} = p^n - \binom{p}{1}(p-1)^n + \binom{p}{2}(p-2)^n + \cdots + (-1)^{p-1}\binom{p}{p-1}.$$
- Application 2: chromatic polynomial in graph coloring problem
- Application 3: Stirling numbers of the second kind
- Application 4: counting derangement

Counting by enumeration

- Enumeration = list all possible outcomes (brute force)
- **Example**: how many ways to form a length-1 string from alphabet $\{1, 2, 3, 4, 5, 5, 6, 7\}$?

$$\mathcal{S} \;=\; \{1, 2, 3, 4, 5, 5, 6, 7\}, \quad |S| = 7.$$

- **Example**: how many ways to form a length-3 string from alphabet $\{0, 1\}$?

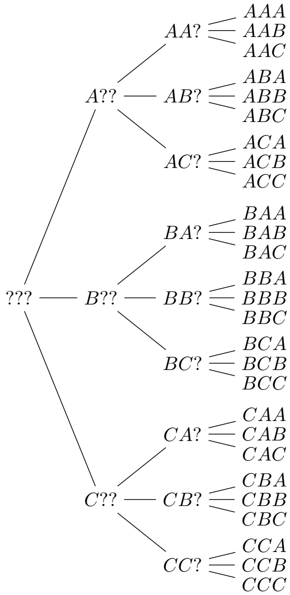
$$\mathcal{S} \;=\; \{000, 001, 010, 100, 011, 101, 110, 111\}, \quad |S| = 8.$$

- **Example**: how many ways to form a length-3 string from alphabet $\{A, B, C\}$?

$$\mathcal{S} \;=\; \{AAA, AAB, ABA, BAA, AAC, ACA, CAA, ABC, BAC, \text{ oh no ...}\}$$

we need a better way to do this.

Tree and multiplication



- Tree = *acyclic connected graph* (details in graph theory)
- **You are CS student, think in a CS way: whenever you count, imagine a tree in your brain!**
- $|\mathcal{S}| = 27 = 3 \cdot 3 \cdot 3$
  - The 3 corresponds to 3 branches
  - The multiplication $\cdot$ corresponds to moving to the next layer. Here we have 3 layer
- Generalize: there are n^k length-k string in a n-set.
- **Example**: how many ways to form a length-4 string from alphabet $\{1, 2, 3, 4, 5, 5, 6, 7\}$?

  $$|S| = 7^4 = 2401.$$

  Before using tree-thinking, you may not even know how to solve this question!

Product rule: direct application of Cartesian product

$$\text{choose } \underbrace{\mathcal{P} = \{A, B, C\}}_{\text{one of these}} \text{ AND } \underbrace{\mathcal{Q} = \begin{Bmatrix} X \\ Y \end{Bmatrix}}_{\text{one of these}} = \text{choose } \underbrace{\mathcal{P} \times \mathcal{Q} = \begin{Bmatrix} AX, BX, CX \\ AY, BY, CY \end{Bmatrix}}_{\text{one of these.}}$$

- $\mathcal{P} \times \mathcal{Q} =$ the Cartesian product of $\mathcal{P}, \mathcal{Q}$
- $|\mathcal{P} \times \mathcal{Q}| = |\mathcal{P}| \cdot |\mathcal{Q}|$
- **Example**. A daily diet consists of a breakfast selected from B, a lunch from L, a dinner from D:

$$\begin{aligned} B &= \{\text{pancakes, bacon and eggs, bagel}\} \\ L &= \{\text{burger and fries, salad, macaroni, pizza}\} \\ D &= \{\text{steak, pasta}\} \end{aligned}$$

  Then the set of all possible daily diets $= B \times L \times D$

  - $\#$possible diets $= |B \times L \times D| \overset{\text{prod. rule}}{=} |B| \cdot |L| \cdot |D| = 3 \cdot 4 \cdot 2 = 24$
  - By using tree: a 3-branch $\times$ 4-branch $\times$ 2-branch tree with 24 edges
  - By enumeration: list all the 24 choice

Product rule, more examples

- **Example** How many length-4 strings over the alphabet $\{0, 1, ..., 9\}$ do not begin with 0?
  - Let A_i be the set of possible alphabet at the ith digit, then

$$\#\text{ strings } = |A_1 \times A_2 \times A_3 \times A_4| \overset{\text{product rule}}{=} |A_1| \cdot |A_2| \cdot |A_3| \cdot |A_4| = 9000$$

  - $|A_1| = 9$ ($\because A_1 = \{1, 2, \ldots, 9\}$)
  - $|A_2| = |A_3| = |A_4| = 10$

  There are 9000 length-4 strings do not begin with 0. This is the #4-digit positive integers with no leading 0.

- **Example** How many length-2 strings over the alphabet $\{0, 1, ..., 9\}$ do not have repeated digits?
  - $|A_1 \times A_2|$
  - $|A_1| = 10$
  - $|A_2| = 9$ choices (1 less than than $|A_1|$ to avoid repetition)

- **Example** How many length-4 strings over the alphabet $\{0, 1, ..., 9\}$ do not have repeated digits?
  - $|A_1 \times A_2 \times A_3 \times A_4| = 10 \cdot 9 \cdot 8 \cdot 7$

Sum rule, a special case of incl-excl principle $|A \cup B| = |A| + |B| - |A \cap B|$

- **Example** In cafe, you have soup xor salad. There are 2 soups, 4 salads on the menu. How many choices do you have?

$$\left|\text{soup} \cup \text{salad}\right| \overset{\text{incl-excl.}}{=} \underbrace{\left|\text{soup}\right| + \left|\text{salad}\right|}_{\text{either soup or salad}} - \underbrace{\left|\text{soup} \cap \text{salad}\right|}_{=0 \ \because \text{ not both}} = 2 + 4 = 6.$$

- **Example** $A = \{1\}, B = \{2, 3\}, C = \{3, 4\}$, what is $|A \cup B \cup C|$?
  - Approach 1: use the inclusion-exclusion principle for 3 sets blindly

$$\begin{aligned}
\left|A \cup B \cup C\right| &\overset{\text{incl-excl.}}{=} |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 1 + 2 + 2 - |\varnothing| - |\varnothing| - |\{3\}| + |\varnothing| \\
&= 1 + 2 + 2 - 0 - 0 - 1 + 0 = 4
\end{aligned}$$

    You need to compute 4 sets: $|A \cap B|, |A \cap C|, |B \cap C|, |A \cap B \cap C|$

  - Approach 2 (smarter)

$$\begin{aligned}
\left|A \cup B \cup C\right| = \left|A \cup (B \cup C)\right| &\overset{\text{incl-excl.}}{=} |A| + |B \cup C| + \left|A \cap (B \cup C)\right| \\
&= 1 + |B| + |C| - |B \cap C| + |\varnothing| \\
&= 1 + 2 + 2 - |\{3\}| + 0 \\
&= 5 - 1 = 4
\end{aligned}$$

    You need to compute 2 sets: $|B \cap C|, |A \cap (B \cup C)|$

  - Approach 3: construct $A \cup B \cup C$.
    This is the most expensive approach if the sets has many elements (you have to check one by one).

Subtraction rule, complement and power set

- Subtraction rule: if $A \subset \mathcal{S}$, then for complement $A^c := \mathcal{S} \setminus A$, we have

$$|A^c| \;=\; |\mathcal{S}| - |A|. \qquad \text{(Subtraction rule)}$$

- **Example** How many subsets of $X = \{1, 2, 3\}$ contain at least 2 (including 2) elements?
  - Let $S \subset X$ that contains at least 2 (including 2) elements.
  - Let $P = 2^X$ be the power set (set of all possible subsets) of X

$$P = \Big\{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\Big\}$$

$$\begin{aligned}
S &= P \setminus S^c \\
&= P \setminus \{\text{set containing 1 or 0 element}\} \\
&= P \setminus \Big(\text{set containing 1 element} \;\cup\; \text{set containing 0 element}\Big) \\
&\overset{\{\}=\varnothing}{=} P \setminus \Big(\{1\} \;\cup\; \{2\} \;\cup\; \{3\} \;\cup\; \{\varnothing\}\Big) \\
|S| &\overset{\text{subtraction rule}}{=} |P| - \Big(|\{1\} \cup \{2\} \cup \{3\}| + |\{\varnothing\}|\Big) \\
&= 2^3 - (3+1) \;=\; 4
\end{aligned}$$

Definition of natural number by Zermelo-Fraenkel set theory NOT in exam

Important note: $\varnothing = \{\}$, and $A = \varnothing, B = \{\varnothing\}$ and $C = \{\{\varnothing\}\}$ are different things. $|A| = 0, |B| = |C| = 1$

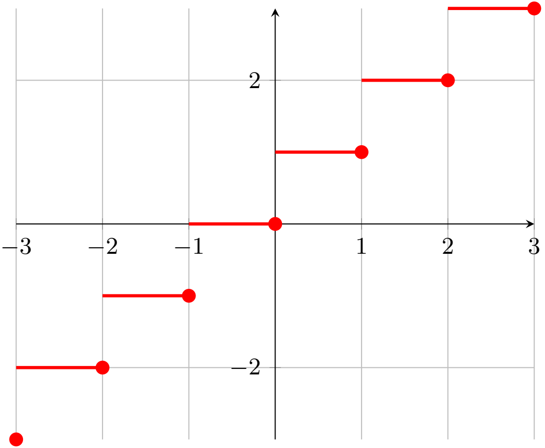
Notation used by John von Neumann.

Number	Set
0	$\lvert\varnothing\rvert$
1	$\lvert\{\varnothing\}\rvert$
2	$\lvert\{\varnothing, \{\varnothing\}\}\rvert$
3	$\left\lvert\left\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\right\}\right\rvert$
4	$\left\lvert\left\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \left\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\right\}\right\}\right\rvert$

Floor function: A $\mathbb{R} \to \mathbb{Z}$ mapping $\lfloor x \rfloor := \max \left\{ m \in \mathbb{Z} \mid m \leq x \right\}$

Example

- $\lfloor 2 \rfloor = 2$
- $\lfloor 2.4 \rfloor = 2$
- $\lfloor 2.00000000001 \rfloor = 2$
- $\lfloor 2.99999999999 \rfloor = 2$
- $\lfloor -2 \rfloor = -2$
- $\lfloor -2.000000001 \rfloor = -3$
- $\lfloor -2.999999999 \rfloor = -3$
- $\lfloor \pi \rfloor = 3$
- $\lfloor -\pi \rfloor = -4$



Ceiling function $\lceil x \rceil := \min \left\{ m \in \mathbb{Z} \mid m \geq x \right\}$.

Example of Incl-excl principle in number theory: #divisible integers

- **Example (AMC12 2001)**[1] How many integers n for $1 \le n \le 2001$ are divisible by 3 or 4 but not 5?

  Ans: 801.

- **Example** $N := \{n \in \mathbb{N} \mid 1 \le n \le 20\}$. How many integers in N are divisible by either 2 or 3?
  - Let A, B be the set of integers in N that are divisible by 2 and by 3, respectively.
  - $|A| = \lfloor \frac{20}{2} \rfloor = 10, \ \ |B| = \lfloor \frac{20}{3} \rfloor = \lfloor 6.666 \rfloor = 6$
  - $A \cap B$ is the set of integers in N that are by 2 AND 3 (=6), and $|A \cap B| = \lfloor \frac{20}{6} \rfloor = \lfloor 3.333 \rfloor = 3$
  - There are $|A \cup B| = |A| + |B| - |A \cap B| = 10 + 6 - 3 = 13$ integers in N that are by 2 or 3.

- **Example** How many integers n where $1 \le n \le 100$ are **NOT** divisible by 2, 3, 5?
  - Let A, B, C be the set of n divisible by 2, by 3, by 5, respectively.
  - Number of integers n divisible by at least one 2, 3, 5 is

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 3} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{100}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 3 \cdot 5} \right\rfloor \\
&= 50 + 33 + 20 - 16 - 10 - 6 + 3 \ = \ 74
\end{aligned}$$

  - By complement, the number of integers n **NOT** divisible by 2, 3, 5 is

$$|\{1, 2, ..., 100\}| - |A \cup B \cup C| = 100 - 74 = 26.$$

[1] American Mathematics Competitions 2001

Product rule and sum rule: counting passwords

- **Example**. A valid password is a sequence of between 6 and 8 symbols. The first symbol must be a letter (lowercase or uppercase), the remaining symbols can be letters or digits. How many different passwords are possible?
- Define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$F = \{a, b, c, \ldots, z, A, B, C, \ldots, Z\}, \qquad S = \{a, b, c, \ldots, z, A, B, C, \ldots, Z, 0, 1, \ldots, 9\}$$

- The Cartesian product set $S^2 := S \times S$ is the set of length-2 strings $\{\alpha\beta \mid \alpha \in S, \beta \in S\}$
- Three possible cases of passwords:

$F \times S^5$	length-6 password
$F \times S^6$	length-7 password
$F \times S^7$	length-8 password

  The set of all possible passwords $(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$, with cardinality

$$\begin{aligned} \left|(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)\right| &= \left|F \times S^5\right| + \left|F \times S^6\right| + \left|F \times S^7\right| && \text{sum rule (incl-excl principle)} \\ &= |F| \cdot |S|^5 + |F| \cdot |S|^6 + |F| \cdot |S|^7 && \text{product rule} \\ &= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 \approxeq 1.8 \times 10^{14} \end{aligned}$$

- 1.8×10^{14} valid passwords. The probability of "correct guess in one trial" is $\dfrac{1}{1.8 \times 10^{14}} = 5 \times 10^{-15}$.

  You are 10^8 times more likely to be hit by thunder.

- If you have 6 chances to guess the password.
  - The probability of all 6 guesses are wrong: $(1 - 5 \times 10^{-15})^6$
  - The probability of at least one guess is correct is $1 - (1 - 5 \times 10^{-15})^6 = 3 \times 10^{-14}$

  You are 10^7 times more likely to be hit by thunder.

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Permutation and factorial

- **Definition** A permutation (denoted by π or σ) is an ordered arrangement of a set of **distinct** objects.
- **Example**: $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and a permutation $\pi(\mathcal{S}) = \{2, 5, 4, 3, 1\}$
  $\pi(1) = 2 =$ the first element in the permuted sequence is 2
  $\pi(3) = 4 =$ the third element in the permuted sequence is 4
- Permutation is not unique.
- **Theorem** The number of permutations of a n-set of distinct objects is $n(n-1) \cdot \dots \cdot 2 \cdot 1$.
  Proof. Tree proof / induction.
- **Example**. How many permutations can be formed from the letters HAPY? $4 \cdot 3 \cdot 2 \cdot 1 = 24$
- **Recursive definition of factorial**

$$k! = \begin{cases} 1 & k = 0 \text{ base case} \\ k(k-1)! & k > 0 \text{ base case} \end{cases} \tag{factorial}$$

  Why $0! = 1$? Google it yourself.

  Not in exam

  - Gamma function generalizes factorial. E.g., $\frac{1}{2}! = \frac{\sqrt{\pi}}{2}$
  - We focus on finite combinatorics (numbers and sets are finite)
  - Crazy things happen at infinity, e.g., $\infty! = 1 \cdot 2 \cdot 3 \cdot \ldots = \sqrt{2\pi}$, this is analytic continuation, like $i = \sqrt{-1}$.

Generalized permutation and division rule: permutation with non-distinct objects

- **Definition** A permutation is an ordered arrangement of a set of **distinct** objects.
- **Definition** A generalized permutation is an ordered arrangement of a set of **non-distinct** objects.
- **Example**. How many permutations can be formed from the letters HAPPY?
  - First, put subscript $\mathsf{HAP_1P_2Y}$ and treat it as 5 distinct letters.
  - There are 5 distinct letters, so $5! = 120$ permutations.
  - But $\mathsf{P_1}$ and $\mathsf{P_2}$ are the same letter, we double-counted some of these 120 permutations. E.g.

$$\mathsf{HAP_1P_2Y}, \quad \mathsf{HAP_2P_1Y} \quad \text{are both HAPPY}$$

$$\mathsf{P_1HAYP_2}, \quad \mathsf{P_2HAYP_1} \quad \text{are both PHAYP}$$

  So correct number of permutations is $\dfrac{120}{2!}$

- **Theorem** The number of permutations of a set of n elements, **possibly non-distinct**, is

$$\frac{n!}{n_1!n_2!\cdots n_r!} \qquad \text{(Division rule)}$$

  where $n_1, n_2, \ldots, n_r$ are number of alike objects.

- If all objects are distinct, $n_1 = n_2 = \cdots = n_r = 1$ and $\dfrac{n!}{n_1!n_2!\cdots n_r!}$ reduces to $n!$.

Division rule example (1/3) **Example** Find the number of permutations of $(1,1,2,3,4)$

- First assume the two 1's are distinct, so we have $5!$ ways to permute $(1_a, 1_b, 2, 3, 4)$
  But in these $5!$ ways, we double-counted some cases:

$$\begin{aligned} &(1,1,2,3,4) \begin{cases} (1_a, 1_b, 2, 3, 4) \\ (1_b, 1_a, 2, 3, 4) \end{cases} \\ &(1,2,1,3,4) \begin{cases} (1_a, 2, 1_b, 3, 4) \\ (1_b, 2, 1_a, 3, 4) \end{cases} \\ &\qquad\qquad\qquad\vdots \end{aligned}$$

  Every $2!$ of our initial $5!$ ways corresponds to one actual permutation. So the $\#$ of permutations of $(1,1,2,3,4)$ is

$$\frac{5!}{2!}$$

- For the formula $\dfrac{n!}{n_1!n_2!\cdots n_r!}$, here
  - $n = 5$: we have 5 things to permute
  - $r = 1$: 1 type of alike objects
  - $n_1 = 2$: for type-1 objects, there are 2 of them

Division rule example (2/3) **Example** Find the number of permutations of $(1, 1, 1, 2, 3)$.

- First assume the three 1's are distinct, so we have $5!$ ways to permute $(1_a, 1_b, 1_c, 2, 3)$
  But in these $5!$ ways, we over-counted some cases:

$$(1,1,1,2,3)\begin{cases}(1_a,1_b,1_c,2,3)\\(1_a,1_c,1_b,2,3)\\(1_b,1_a,1_c,2,3)\\(1_b,1_c,1_a,2,3)\\(1_c,1_a,1_b,2,3)\\(1_c,1_b,1_a,2,3)\end{cases}$$

$$\vdots$$

  Every $3! = 6$ of our initial $5!$ ways corresponds to one actual permutation So the $\#$ of permutations of $(1, 1, 1, 2, 3)$ is

$$\frac{5!}{3!}$$

- For the formula $\dfrac{n!}{n_1!n_2!\cdots n_r!}$, here
  - $n = 5$: we have 5 things to permute
  - $r = 1$: 1 type of alike objects
  - $n_1 = 3$: for type-1 objects, there are 3 of them

Division rule example (3/3) **Example** Find the number of permutations of $(1, 1, 1, 2, 2)$

- First assume the three 1's and the two 2's are distinct, so we have $5!$ ways to permute $(1_a, 1_b, 1_c, 2_a, 2_b)$
  But in these $5!$ ways, we over-counted some cases:

$$(1,1,1,2,2)\begin{cases}(1_a,1_b,1_c,2_a,2_b)\\(1_a,1_c,1_b,2_a,2_b)\\(1_b,1_a,1_c,2_a,2_b)\\(1_b,1_c,1_a,2_a,2_b)\\(1_c,1_a,1_b,2_a,2_b)\\(1_c,1_b,1_a,2_a,2_b)\\(1_a,1_b,1_c,2_b,2_a)\\(1_a,1_c,1_b,2_b,2_a)\\(1_b,1_a,1_c,2_b,2_a)\\(1_b,1_c,1_a,2_b,2_a)\\(1_c,1_a,1_b,2_b,2_a)\\(1_c,1_b,1_a,2_b,2_a)\end{cases}$$

$$\vdots$$

  Every $3!2! = 12$ of our initial $5!$ ways corresponds to one actual permutation.

  So the # of permutations of $(1, 1, 1, 2, 3)$ is $\dfrac{5!}{3!2!}$

Now you understand what is $\frac{n!}{n_1!n_2!\cdots n_r!}$

$$\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{\text{(number of objects)!}}{\text{(number of type-1 objects)!(number of type-2 objects)!}\cdots\text{(number of type-r objects)!}}$$

- $n_1 + n_2 + \cdots + n_r \leq n$
  It is impossible for the sum $n_1 + n_2 + \cdots + n_r$ to exceed n.

- In fact, such expression is known as the **multinomial coefficient**

$$\frac{n!}{n_1!n_2!\cdots n_r!} =: \binom{n}{n_1, n_2, \ldots, n_r} \qquad \text{(multinomial coefficient)}$$

- **Example** How many permutation for the string "`baby`"?
  - type-1 letter `b` has 2 occurrence: divided by $2!$
  - type-2 letter `a` has 1 occurrence: divided by $1!$
  - type-3 letter `y` has 1 occurrence: divided by $1!$
  - $r = 3$ and $n = 4$ here
  - Solution $\frac{4!}{2!1!1!} = 12$

$\frac{n!}{n_1!n_2!\cdots n_r!=}$ #permutations of a n-set with repeated elements is

- **Example**. A chess tournaments has 10 competitors, of which

  4 are from Russia   3 are from US   2 are from UK   1 is from Brazil

  If the tournament result lists the nationalities of the players in the order in which they placed, how many outcomes are possible?

  - 10 competitors, so $10!$ permutations
  - 4 Russians, so $4!$ repeated permutations to be divided
  - 3 US, so $3!$ repeated permutations to be divided
  - 2 UK, so $2!$ repeated permutations to be divided
  - 1 Brazil, so $1!$ repeated permutations to be divided

$$\frac{10!}{4!3!2!1!} = 12600$$

Derangement Not in exam

- **Definition**. Derangement = permutation that no element appears in its original position.
- For $\{1,2,3,4\}$, the number of derangement is 9.
- The number of derangement, denoted by $!n$, has the formula

$$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

  How do you prove this: Incl-excl principle.

- Example.

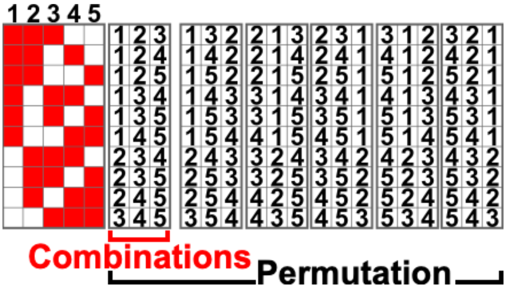
$$!4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9.$$

1234	2314	3412
1243	2341	3421
1324	2413	4123
1342	2431	4132
1423	3124	4213
1432	3142	4231
2134	3214	3412
2143	3241	4321

Combination

- Combination vs Permutation = commutative vs not commutative
  - **Definition** An relation R is commutative if $aRb = bRa$.
  - A permutation is an **ordered** arrangement $\quad aab \neq aba \neq baa$
  - A combination is an **unordered** arrangement $\quad aab = aba = baa$
  - Boxing analogy: you fight Elon Musk, Joe Biden, Mike Tyson
    - Permutation: the order (Mike Tyson, Elon Musk, Joe Biden) $\neq$ (Joe Biden, Elon Musk, Mike Tyson)
    - Combination: the order (Mike Tyson, Elon Musk, Joe Biden) = (Joe Biden, Elon Musk, Mike Tyson)

- **Example** A committee of 3 is to be formed from a group of 5 people $(1, 2, 3, 4, 5)$. How many different committees are possible?



Binomial coefficient & number of combinations

- **Definition** Given integers $n \geq k \geq 0$, the n-choose-k, denoted as $\binom{n}{k}$, is defined as

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}. \qquad \text{(Binomial coefficient n-choose-k)}$$

- **Example**: 5-choose-3 is

$$\binom{5}{3} = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$$

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

- Meaning of binomial coefficients
  - number of combinations in n-choose-k / number of k-sets in n-set
  - number of bit strings from length n with exactly k 1's
  - the coefficient of $x^k y^{n-k}$ in $(x+y)^n$

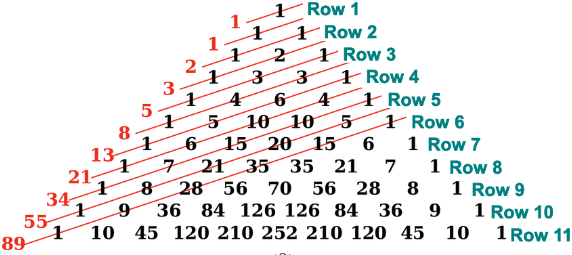
- If $k < 0$ or $k > n$, $\binom{n}{k}$ is meaningless and defined as 0.

	power-1 term	power-2 term	power-3 term	power-4 term
$(1+x)^1$	1			
$(1+x)^2$	2	1		
$(1+x)^3$	3	3	1	
$(1+x)^4$	4	6	4	1

is the same as

	power-1 term	power-2 term	power-3 term	power-4 term
$(1+x)^1$	$\binom{1}{1}$			
$(1+x)^2$	$\binom{2}{1}$	$\binom{2}{2}$		
$(1+x)^3$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$	
$(1+x)^4$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$

Pascal's Δ, Fibonacci sequence, binomial coefficient



$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}
\end{array}
$$

The figures tell many things

Properties of binomial coefficient from Pascal's Δ

- $\binom{n}{0} = \binom{n}{n} = 1$ the outermost columns in Pascal's Δ

- $\binom{n}{k} = \binom{n}{n-k}$ horizontal symmetry in Pascal's Δ

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ sum of elements in the nth row

  also the number of subsets of a n-element set

- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ equivalently $\underbrace{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}_{\text{a recurrence relation}}$ sum of consecutive

  (Can you see that $\uparrow$ is a sum rule?)

- $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$ sum of half of Pascal's Δ is the next Fibonacci number

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$,	integers $n \geqslant k \geqslant 0$.	*factorial expansion*
$\binom{n}{k} = \binom{n}{n-k}$,	integer $n \geqslant 0$, integer k.	*symmetry*
$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$,	integer $k \neq 0$.	*absorption/extraction*
$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$,	integer k.	*addition/induction*
$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$,	integer k.	*upper negation*
$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$,	integers m, k.	*trinomial revision*
$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r$,	integer $r \geqslant 0$, or $\lvert x/y \rvert < 1$.	*binomial theorem*
$\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$,	integer n.	*parallel summation*
$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$,	integers $m, n \geqslant 0$.	*upper summation*
$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,	integer n.	*Vandermonde convolution*

← Table 174
Concrete mathematics: a foundation for computer science
by Graham, Knuth and Patashnik.
The wiki page

Proving $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$

Algebraic proof

$$
\begin{aligned}
& \binom{n}{k}+\binom{n}{k+1} \\
= \quad & \frac{n!}{(n-k)!k!}+\frac{n!}{(n-k-1)!(k+1)!} \\
= \quad & \frac{n!}{(n-k)!k!}\frac{k+1}{k+1}+\frac{n!}{(n-k-1)!(k+1)!}\frac{n-k}{n-k} \\
= \quad & \frac{n!(k+1)}{(n-k)!(k+1)!}+\frac{n!(n-k)}{(n-k)!(k+1)!} \\
= \quad & \frac{n!\Big(k+1+n-k\Big)}{(n-k)!(k+1)!} \\
= \quad & \frac{(n+1)!}{((n+1)-(k+1))!(k+1)!} \\
= \quad & \binom{n+1}{k+1}
\end{aligned}
$$

Algebraic proof can be tedious.
Thus we should consider combinatorial proof.

Combinatorial proof (Not in exam)

- Find an appropriate object to be counted
- Count that object in some way
- Count that object in a different way

Then the two counting methods give the same result.

The proof Let $S=\{a,b,c,\cdots,z\}$ has $n+1$ people.

- (1st counting) There are $\binom{n+1}{k+1}$ ways to pick $k+1$ people in S.
- (2nd counting) Suppose e is a VIP in S. Now, we we are doing the same thing: "selecting the $k+1$ people in S". With respect to e, there are two possibilities.
  - VIP is selected. Then we know we need to pick k more people (not $k+1$ because e has been selected here) in the remaining n people (not $n+1$ because e has been selected). There are $\binom{n}{k}$ ways.
  - VIP is not selected. Then we need to pick $k+1$ (because e is not selected so we still need to pick $k+1$ people) people in the remaining n (not $n+1$ because e has been rejected) people. There are $\binom{n}{k+1}$ ways.

  These two yes-no events are disjoint so $\binom{n}{k}+\binom{n}{k+1}$.
- Both counting method 1 and method 2 are counting the same people, so
$$\binom{n+1}{k+1}=\binom{n}{k}+\binom{n}{k+1}.$$

Binomial coefficient for counting combinations: no repetition

- $\binom{n}{k}$ ways to choose k elements from a n-set if repetitions of elements are NOT allowed

- **Example** Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.
  - How many subsets are there total? $2^6 = 64$
  - How many 4-subsets are there? 6-choose-4, $\binom{6}{4}$

- **Example** How many length-10 strings over alphabet $\{0, 1\}$ contain 6 or more 1's?
  - Let S_6 be the set of strings containing six 1's
  - We need to compute $\left|S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_9 \cup S_{10}\right|$, by inclusion-exclusion principle

$$\begin{aligned}
\left|S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_9 \cup S_{10}\right| &\overset{\text{sum rule}}{=} |S_6| + |S_7| + |S_8| + |S_9| + |S_{10}| \\
&= \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \\
&= 386
\end{aligned}$$

$\binom{n+k-1}{k}$ Binomial coefficient for counting combinations: with repetition ... (1/2)

$\binom{n+k-1}{k}$ ways to choose k elements from a n-set if repetitions of elements are allowed

- **Example** A farm has cats and dogs. How many ways can you select three pets to take home?
  - $F = \{1, 2\}$, 1=cat, 2=dog.
  - Let's enumerate all the $2^3 = 8$ possible cases

$$\begin{array}{cccc} 111 & 112 & 121 & 122 \\ 211 & 212 & 221 & 222 \end{array} \quad 2^3 = 8 \text{ cases}$$

  - There are repetitions within the combination: e.g. $\underline{1}\,\underline{1}\,2$ and $\underline{2}\,\underline{2}\,\underline{2}$
  - Combination is **unordered**, so $112 = 211$. After removing the same combinations we have

$$\begin{array}{ccc} 111 & 112 & 122 \\ & & 222 \end{array} \quad 4 \text{ cases} \qquad 4 = \binom{2+3-1}{3} = \binom{4}{3}.$$

- **Example** A farm has birds, cats and dogs. How many ways can you select three pets to take home?
  - $F = \{1, 2, 3\}$, 1=bird, 2=cat, 3 =dog. Enumerate all the $3^3 = 27$ possible cases

$$\begin{array}{ccccccccc} 111 & 112 & 113 & 121 & 122 & 123 & 131 & 132 & 133 \\ 211 & 212 & 213 & 221 & 222 & 223 & 231 & 232 & 233 \\ 311 & 312 & 313 & 321 & 322 & 323 & 331 & 332 & 333 \end{array} \quad 27 \text{ cases}$$

  - There are **repetitions**: e.g. $\underline{1}\,\underline{1}\,2$ and $\underline{2}\,\underline{2}\,3$
  - Combination is **unordered**, so $133 = 313 = 331$. After removing the same combinations we have

$$\begin{array}{cccccc} 111 & 112 & 113 & 122 & 123 & 133 \\ & & & 222 & 223 & 233 \\ & & & & & 333 \end{array} \quad 10 \text{ cases} \qquad 10 = \binom{3+3-1}{3} = \binom{5}{3}.$$

$\binom{n+k-1}{k}$ Binomial coefficient for counting combinations: with repetition ... (2/2)

- **Example** A farm has birds, cats, dogs, ducks. How many ways can you select three pets?
  - $F = \{1, 2, 3, 4\}$, 1=bird, 2=cat, 3 =dog, 4 =duck. Enumerate all the $4^3 = 64$ possible cases

$$\begin{array}{cccccc} 111 & 112 & 113 & 114 & 121 & \cdots \\ 211 & 212 & 213 & 214 & 221 & \cdots \\ 311 & 312 & 313 & 324 & 321 & \cdots \\ 411 & 412 & 413 & 424 & 421 & \cdots \end{array} \quad 64 \text{ cases}$$

  - There are **repetitions**: e.g. $\underline{1}\,\underline{1}\,2$ and $\underline{2}\,\underline{2}\,3$
  - Combination is **unordered**, so $133 = 313 = 331$. After removing the same combinations we have

$$\binom{4+3-1}{3} = \binom{6}{3} = 20.$$

- **Example** There are five types of batteries: AAA, AA, C, D, and 9-volt. How many ways can we choose the twenty batteries?

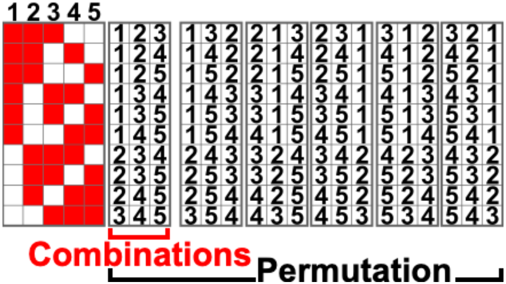
$$\binom{5+20-1}{20} = \binom{24}{20} = 10626.$$

- **Example** There are 5 types of soda. You buy 7 cans of soda. How many selections can you make?

$$\binom{5+7-1}{7} = \binom{11}{7} = 330.$$

Combination and permutation together, example .. (1/2)

- **Example** In boxing, you fight against 3 boxers consecutively from a group $(1, 2, 3, 4, 5)$. How many different fight orders are possible?



- By combination: you pick 3 out of 5, so 5-choose-3, $\binom{5}{3}$ possible groups.
- In each group, there are $3!$ permutation, so by product rule, the total number of fight orders

$$\binom{5}{3}3!$$

- **Generalization** The number of permutation of k subset of a n-set is $\binom{n}{k}k!$.

  Some books denote this as $P(n,k) = \dfrac{n!}{(n-k)!}$

Combination and permutation together, example .. (2/2)

- **Example** From the 26-alphabet, how many length-5 strings, no repeated letters are allowed, are possible?
  $$\binom{26}{5}5!$$
- **Example** From the 26-alphabet, how many length-5 strings, with repeated letters are allowed, are possible?
  $$\binom{26}{5}5^5$$
- **Example** From the 26-alphabet, how many length-5 strings, with the first two letters cannot be the same, are possible?
  $$\binom{26}{5}5 \cdot 4 \cdot 5 \cdot 5 \cdot 5 = \binom{26}{5} \cdot 4 \cdot 5^4$$
- **Example** From the 26-alphabet, how many length-7 English names are possible ?
  $$\binom{26}{7}7^7$$
- **Example** From the 26-alphabet, how many length-7 English names start with letter a are possible ?
  $$\binom{26}{7}7^7 - 25 \cdot 26^6 = 534003691000$$

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- **Example**. $(x+y)^3$

$$\begin{aligned} & (x+y)^3 \\ = \; & \binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0 \\ = \; & y^3 + 3xy^2 + 3x^2y + x^3. \end{aligned}$$

- **Notation** $[x^a]f(x)$ means the coefficient of x^a in $f(x)$
- **Example**. $[x^5](1+2x)^9 = \binom{9}{5} \cdot 2^5$.
- **Example**. $[x^5y^6](5x-3y)^{11} = \binom{11}{5} \cdot 5^5 \cdot (-3)^6$.
- **Example**. $5^n = (3+2)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k}$

- To find the coefficient of $x^p y^q$, find the k $(1 \le k \le n, k \in \mathbb{N})$ in the binomial expansion that give $x^p y^q$. If such k doesn't exist, there is no $x^p y^q$ in the expansion.
- **Example**. The coefficient of x^3y^3 in $(x^2+y^2)^3$ is 0.

  Proof. Let $c = [x^3y^3](x^2+y^2)^3$. By binomial expansion

$$cx^3y^3 = \underbrace{\binom{3}{k}\left(x^2\right)^k\left(y^2\right)^{3-k}}_{k\text{th term in the expansion}} = \binom{3}{k}x^{2k}y^{6-2k}.$$

  Now comparing the indices:

  Find the integer k that solves $3 = 2k$ AND $3 - k = 6 - 2k$

  There is no solution, thus x^3y^3 does not exist in $(x^2+y^2)^3$ and $c = 0$.

Mathematical induction NOT in exam 2023

- To prove a statement S_n, for n starting from 1,

  Simple induction

  - **Base case** Prove S_1 is true.
  - **Hypothesis step** Assume S_m is true for some m
  - **Induction step** Using S_m to prove S_{m+1} is true.

  Strong induction not in exam

  - **Base case** Prove S_1 and S_2 are true.
  - **Hypothesis step** Assume S_m, S_{m+1} are true for some m
  - **Induction step** Using S_m and S_{m+1} to prove S_{m+2} is true.

  Generalized strong induction not in exam

  - **Base case** Prove $S_1, S_2, \ldots, S_k$ are true.
  - **Hypothesis step** Assume $S_m, S_{m+1}, ..., S_{m+k-1}$ are true for some m
  - **Induction step** Using $S_m, S_{m+1}, ...S_{m+k-1}$ to prove S_{m+k} is true.

- Related concept: well-ordering principle not in exam

Proving the binomial theorem via mathematical induction $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

- We will need identities $\binom{p}{0} = \binom{q}{0}$ and $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
- **Proof** In three steps.
  Base case Consider case $n = 1$, we have $(x+y)^1 = x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$.
  Hypothesis step Assume the statement at $n = m$

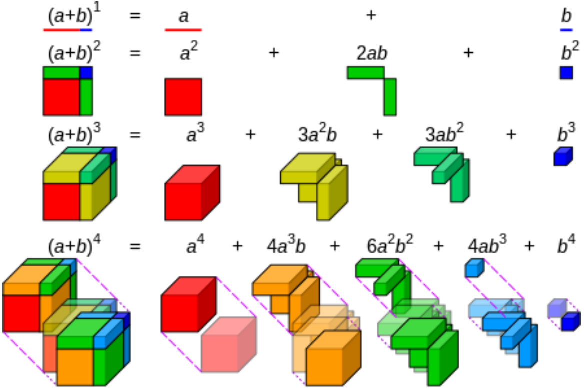
$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \tag{$\mathcal{H}$}$$

  Induction step Consider the statement at $m+1$, we have

$$\begin{aligned}
(x+y)^{m+1} &= (x+y)(x+y)^m \\
&\overset{(\mathcal{H})}{=} (x+y)\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\
&= (x+y)\left(\binom{m}{0}x^0y^m + \binom{m}{1}x^1y^{m-1} + \cdots + \binom{m}{m-1}x^{m-1}y^1 + \binom{m}{m}x^my^0\right) \\
&= \begin{array}{l} \binom{m}{0}x^1y^m + \binom{m}{1}x^2y^{m-1} + \cdots + \binom{m}{m-1}x^my^1 + \binom{m}{m}x^{m+1}y^0 \\ +\binom{m}{0}x^0y^{m+1} + \binom{m}{1}x^1y^m + \cdots + \binom{m}{m-1}x^{m-1}y^2 + \binom{m}{m}x^my^1 \end{array} \\
&= \binom{m}{0}x^0y^{m+1} + \left(\binom{m}{0} + \binom{m}{1}\right)x^1y^m + \left(\binom{m}{1} + \binom{m}{2}\right)x^2y^{m-1} + \cdots + \left(\binom{m}{m-1} + \binom{m}{m}\right)x^my^1 + \binom{m}{m}x^{m+1}y^0 \\
&= \binom{m+1}{0}x^0y^{m+1} + \binom{m+1}{1}x^1y^m + \binom{m+1}{2}x^2y^{m-1} + \cdots + \binom{m}{m}x^my^1 + \binom{m+1}{m+1}x^{m+1}y^0 \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m+1-k} \quad \square
\end{aligned}$$

Picture of binomial expansion.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1



Picture of trinomial expansion $(x+y+z)^3 = x^3+y^3+z^3+$cross-terms

$$x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + 3y^2z + 3yz^2$$

Trinomial expansion

$$\boxed{(p+q)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}}$$

- Cast trinomial as binomial

$$(x+y+z)^n \;=\; \Big(x+(y+z)\Big)^n \overset{\text{bi.thm}}{=} \sum_{k=0}^{n} \binom{n}{k} x^k (y+z)^{n-k} \overset{\text{bi.thm}}{=} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{n-k} \binom{n-k}{j} y^j z^{n-k-j}.$$

- Combine the binomial coefficients

$$\binom{n}{k}\binom{n-k}{j} = \frac{n!}{(n-k)!k!}\frac{(n-k)!}{(n-k-j)!j!} = \frac{n!}{k!j!(n-k-j)!}$$

Rename $n-k-j=i$

$$(x+y+z)^n \;=\; \sum_{\substack{i,j,k\\ i+j+k=n}} \frac{n!}{i!j!k!} x^k y^j z^i \;=\; \sum_{\substack{i,j,k\\ i+j+k=n}} \frac{n!}{i!j!k!} x^i y^j z^k.$$

$\displaystyle\binom{n}{i,j,k} := \frac{n!}{i!j!k!}$ with $i+j+k=n$ is the trinomial coefficient.

Trinomial expansion example: Expand $(x+y+z)^4$

- $\frac{4!}{4!0!0!}x^4$

- $\frac{4!}{3!1!0!}x^3y^1$, $\frac{4!}{3!0!1!}x^3z^1$

- $\frac{4!}{2!2!0!}x^2y^2$, $\frac{4!}{2!1!1!}x^2y^1z^1$, $\frac{4!}{2!0!2!}x^2z^2$

- $\frac{4!}{1!3!0!}x^1y^3$, $\frac{4!}{1!2!1!}x^1y^2z^1$, $\frac{4!}{1!1!2!}x^1y^1z^2$, $\frac{4!}{1!0!3!}x^1y^0z^3$

- $\frac{4!}{0!4!0!}y^4$, $\frac{4!}{0!3!1!}y^3z^1$, $\frac{4!}{0!2!2!}y^2z^2$, $\frac{4!}{0!1!3!}z^1z^3$, $\frac{4!}{0!0!4!}z^4$,

$$x^4+4x^3y+4x^3z+6x^2y^2+12x^2yz+6x^2z^2+4xy^3+12xy^2z+12xyz^2+4xz^3+y^4+4y^3z+6y^2z^2+4yz^3+z^4$$

- How many terms: 3-choose-4 with repetition

- Those coefficients are in Newton polytope

Multinomial expansion

$$\left(x_1 + \cdots + x_m\right)^n = \sum_{\substack{k_1,\ldots,k_m \geq 0 \\ k_1+\cdots+k_m=n}} \binom{n}{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m}.$$

Example $(x+y+z+w)^4$

- $\frac{4!}{4!0!0!0!}x^4$
- $\frac{4!}{3!1!0!0!}x^3y^1$, $\frac{4!}{3!0!1!0!}x^3z^1$, $\frac{4!}{3!0!0!1!}x^3w^1$
- $\frac{4!}{2!2!0!0!}x^2y^2$, $\frac{4!}{2!1!1!0!}x^2y^1z^1$, $\frac{4!}{2!1!0!1!}x^2y^1w^1$, $\frac{4!}{2!0!2!0!}x^2z^2$, $\frac{4!}{2!0!1!1!}x^2z^1w^1$, $\frac{4!}{2!0!0!2!}x^2w^2$,
- $\frac{4!}{1!3!0!0!}x^1y^3$, $\frac{4!}{1!2!1!0!}x^1y^2z^1$, $\frac{4!}{1!2!0!1!}x^1y^2w^1$ and so on ...

SUPER IMPORTANT TABLE

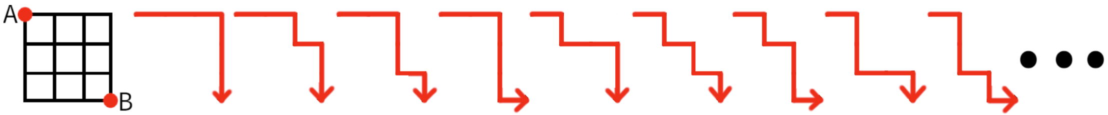
	order	no order
no repetition	n-permutation $n!$	n-choose-k combination $\binom{n}{k}$
repetition	generalized permutation $\binom{n}{n_1, n_2 \ldots, n_r}$	generalized combination $\binom{n+k-1}{k}$

You think these are all the special numbers? Here are more (Not in exam)

- Falling factorial power: $k^{\underline{n}} = \dfrac{k!}{(k-n)!}$
- Rising factorial power: $k^{\overline{n}} = \dfrac{(k+n-1)!}{(k-1)!}$
- Stirling numbers of the first kind: $\begin{bmatrix} n \\ k \end{bmatrix} = \#$ of ways of seating n people around k identical non-empty circular tables
- Stirling numbers of the second kind: $\left\{ {n \atop k} \right\}$ $\#$ of ways of putting n distinct balls into k identical boxes with at least one ball per box

Lattice path

- How many **shortest** lattice path start at $(0,3)$ and end at $(3,0)$?
- Lattice = we are on a grid with lattice points
- Analysis: draw a grid



  - To reach B from A in a **shortest** path, we can only move South (S) or East (E).
  - Moving from A to B has 6 steps, e.g., EEESSS
  - EEESSS is the same as "in 6 boxes, put 3 E and 3 S"
  - So the question is the same as "how many ways to put 3 E and 3 S in 6 boxes".

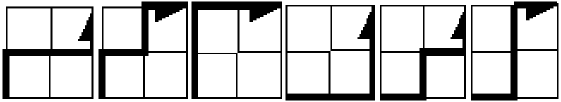
$$\binom{6}{3}$$

Lattice path examples

- **Theorem** In a 2D plane, the number of shortest lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$

- **Example** From $(0,0)$ to $(1,1)$, there are $\binom{1+1}{1} = 2$ shortest lattice paths with step 2.

- **Example** From $(0,0)$ to $(2,2)$, there are $\binom{2+2}{2} = 6$ shortest lattice paths with step 4.



- **Example** From $(2,2)$ to $(3,4)$, there are $\binom{1+2}{1} = 3$ shortest lattice paths with step 3.

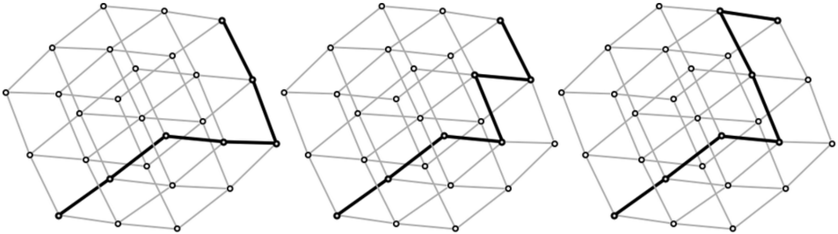
Example How many lattice paths from $(0,0)$ to $(10,10)$ that pass through $(3,3)$ but not pass $(5,5)$?

- Product rule: consider $\Big\{(0,0) \to (3,3)\Big\} \times \Big\{(3,3) \to (10,10) \text{ that do not pass through } (5,5)\Big\}$
- $\Big\{(0,0) \to (3,3)\Big\}$ has $\binom{3+3}{3}$ possibilities.
- $\Big\{(3,3) \to (10,10) \text{ that do not pass through } (5,5)\Big\} = \Big\{(0,0) \to (7,7) \text{ that do not pass through } (2,2)\Big\}$
- Complement rule: $\Big\{(0,0) \to (7,7) \text{ that do not pass through } (2,2)\Big\} = \text{all paths} - \Big\{(0,0) \to (7,7) \text{ that pass through } (2,2)\Big\}$
- Treat $\Big\{(0,0) \to (7,7) \text{ that pass through } (2,2)\Big\}$ using product rule again
- Solution:

$$
\begin{aligned}
& \left|\Big\{(0,0) \to (3,3)\Big\} \times \Big\{(3,3) \to (10,10) \text{ that do not pass through } (5,5)\Big\}\right| \\
\overset{\text{prod. rule}}{=} \; & \left|\Big\{(0,0) \to (3,3)\Big\}\right| \cdot \left|\Big\{(0,0) \to (7,7) \text{ that do not pass through } (2,2)\Big\}\right| \\
\overset{\text{complement}}{=} \; & \tbinom{3+3}{3}\left(\left|\Big\{(0,0) \to (7,7)\Big\}\right| - \left|\Big\{(0,0) \to (7,7) \text{ that pass through } (2,2)\Big\}\right|\right) \\
= \; & \tbinom{6}{3}\left(\tbinom{7+7}{7} - \left|\Big\{(0,0) \to (7,7) \text{ that pass through } (2,2)\Big\}\right|\right) \\
\overset{\text{prod. rule}}{=} \; & \tbinom{6}{3}\left(\tbinom{14}{7} - \left|\Big\{\{(0,0) \to (2,2)\} \times \{(2,2) \to (7,7)\}\Big\}\right|\right) \\
\overset{\text{prod. rule}}{=} \; & \tbinom{6}{3}\left(\tbinom{14}{7} - \left|\{(0,0) \to (2,2)\}\right| \cdot \left|\{(2,2) \to (7,7)\}\right|\right) \\
\overset{\text{prod. rule}}{=} \; & \tbinom{6}{3}\left(\tbinom{14}{7} - \tbinom{2+2}{2}\tbinom{5+5}{5}\right) \\
= \; & \tbinom{6}{3}\left(\tbinom{14}{7} - \tbinom{4}{2}\tbinom{10}{5}\right)
\end{aligned}
$$

Lattice path and multinomial coefficient

- **Theorem** In a 2D plane, the number of lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$
- **Theorem** In a 3D cube, the number of lattice paths from $(0,0,0)$ to (m,n,p) is $\binom{m+n+p}{m,n,p}$



(0,0,0) to (2,2,2) has $\binom{2+2+2}{2,2,2} = \frac{(2+2+2)!}{2!2!2!} = \frac{6!}{8} = 90$ lattice paths

- How many lattice paths from $(0,1,2)$ to $(5,5,5)$ that pass through $(3,3,3)$ but not $(1,2,3)$?
- **Generalization** In a n-dimensional hypercube, the number of lattice paths from $(0,0,...,0)$ to $(k_1,k_2,...,k_n)$ is $\binom{k_1+...+k_n}{k_1,...,k_n}$.

Table of Contents

Rising number

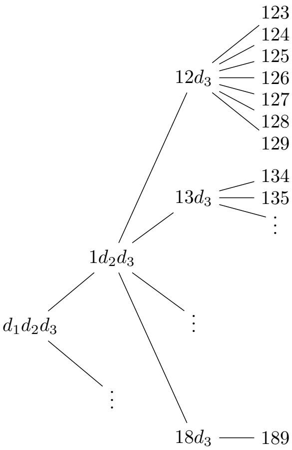
- **Definition** A positive integer is called a *rising number* if its digits form a strictly increasing sequence.

$$\text{Rising number} := \Big\{ d_1 d_2 \cdots d_n \mid d_i \in \{1, ..., 9\},\ d_1 < d_2 < \cdots < d_n \Big\}.$$

Number	is rising?
123	yes
343	no
2334	no
1	yes

- How many 3-digit rising numbers are there?
  Let
  $$\mathcal{S} := \Big\{ d_1 d_2 d_3 \,|\, d_i \in \{1, ..., 9\},\ d_1 < d_2 < d_3 \Big\}.$$
  The question is asking: what is $|\mathcal{S}|$?

Counting rising number via enumeration, product rule and sum rule



- Case $d_1 = 1$
  - If $d_2 = 2$, we can take $d_3 \in \{3, 4, 5, 6, 7, 8, 9\}$ — 7 possibilities
  - If $d_2 = 3$, we can take $d_3 \in \{4, 5, 6, 7, 8, 9\}$ — 6 possibilities
  - If $d_2 = 4$, we can take $d_3 \in \{5, 6, 7, 8, 9\}$ — 5 possibilities
  - so on, we have $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$
- Case $d_1 = 2$
  - If $d_2 = 3$, we can take $d_3 \in \{4, 5, 6, 7, 8, 9\}$ — 6 possibilities
  - If $d_2 = 4$, we can take $d_3 \in \{5, 6, 7, 8, 9\}$ — 5 possibilities
  - If $d_2 = 5$, we can take $d_3 \in \{6, 7, 8, 9\}$ — 4 possibilities
  - so on, we have $6 + 5 + 4 + 3 + 2 + 1 = 21$
- and so on for $d_1 = 3$ to $d_1 = 6$ — 15, 10, 6, 3
- Case $d_1 = 7$
  - If we take $d_2 = 8$, we can only take $d_3 = 1$
  - If we take $d_2 = 9$, no number is possible
- Case $d_1 = 8$, we can only take $d_2 = 9$ but no number is possible for d_3
- Case $d_1 = 9$, no number is possible.

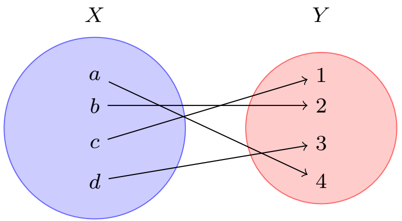
  Total number of cases: 28 + 21 + 15 + 10 + 6 + 3 + 1 = 84

Counting via bijection

- Assume we have a bijection from set X to set Y
- "If counting on X is hard, count on Y instead"
- Counting via bijection is a powerful technique.
- **Example**

  $$[x^r](1+x)^n \quad \text{and} \quad \#\text{ of } r\text{-combinations of } n\text{-set}$$

  is a bijection.
  This is why we can use binomial coefficient to count things.



Bijection: one-to-one (injective) and onto (surjective)

Counting rising number by bijection

- Fact 1: $d_1 < d_2 < d_3$ (rising number) $\implies$ $d_1 \neq d_2 \neq d_3$ (3 distinct digits)
  Fact 2: $d_1 \neq d_2 \neq d_3$ (3 distinct digits) $\implies$ there is exactly 1 rising number.
  e.g. for $\{4, 7, 1\}$, the raising number is 147 and there is only one possible raising number.

Hence there is a bijection between

# of 3-digit rising numbers and # of ways of selecting 3 distinct digits

# of ways of selecting 3 distinct digits from 9 symbols = 9-choose-3 $= \binom{9}{3} = \frac{9!}{6!3!} = 84.$

Examples of rising numbers

- **Example** How many 4-digit rising numbers are there that are greater than 5000?
  - $>$5000 means we are limited to $\{5, 6, 7, 8, 9\}$
  - By the bijection argument, the number is 5-choose-4 $\binom{5}{4} = 5.$

  Or you can choose to list them

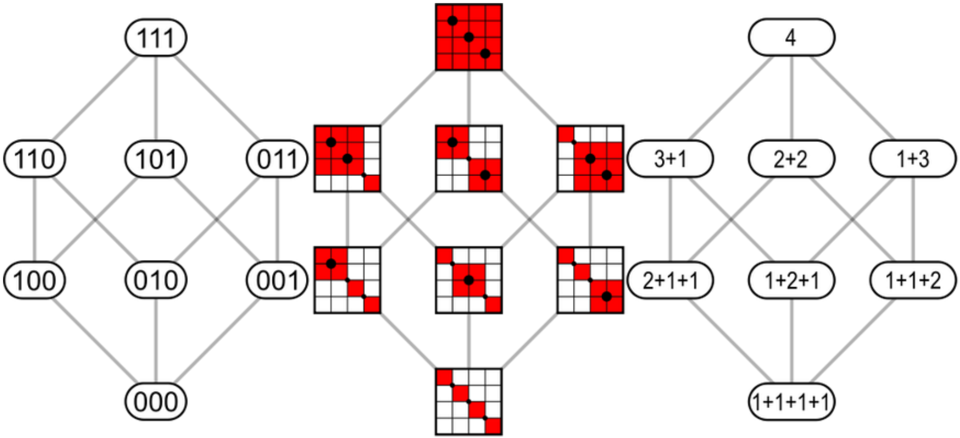
$$5678,\ 5679,\ 5689,\ 5789,\ 6789$$

- **Example** How many 4-digit rising numbers with $d_4 = 9$ are there?
  - d_4 is fixed so we do not need to consider d_4
  - This problem is the same as counting how many 3-digit rising numbers with $d_i \in \{1, 2, ..., 8\}$

$$\binom{8}{3}.$$

- **Example** With alphabet $\{a, b, c, ..., z\}$, how many 5-letter rising strings with $d_2 = d$ and $d_3 = f$ are there?
  - This problem is the same as counting how many rising strings as "$?df??$"
  - The first "$?$" has 3 choices: $\{a, b, c\}$
  - The last two "$??$" have $\binom{20}{2}$: choosing $\{g, h, i, ..., z\}$ for 2 box
  - So the total number is $3\binom{20}{2} = 570$

Bijection: composition of number, bit strings, tildes



Bijection between 3-bit, tildes and compositions of 4

- **Composition** of a integer means writing this number as a sum of other distinct integers.
- **Example** $5 = 1 + 4$ is a composition, $5 = 1 + 1 + 1 + 2$ is also a composition.

Twelvefold way of balls and boxes

f-class	any f (no rules on placement)	Injective f (no multi-packs allowed)	Surjective f (no empty box allowed)
Distant f	How many ways can you place n marked balls into x marked boxes, with no other rules on placement? x^n	How many ways can you place n marked balls into x marked boxes, with no multi-packs allowed? $x(x-1)\ldots(x-n+1)$	How many ways can you place n marked balls into x marked boxes, with no empty boxes allowed? out of scope
S_n orbit $f \circ S_n$	How many ways can you place n plain balls into x marked boxes, with no other rules on placement? $\binom{x+n-1}{n}$	How many ways can you place n plain balls into x marked boxes, with no multi-packs allowed? $\binom{x}{n}$	How many ways can you place n plain balls into x marked boxes, with no empty boxes allowed? $\binom{n-1}{n-x}$
S_x orbit $S_x \circ f$	How many ways can you place n marked balls into x plain boxes, with no other rules on placement? out of scope	How many ways can you place n marked balls into x plain boxes, with no multi-packs allowed? out of scope	How many ways can you place n marked balls into x plain boxes, with no empty boxes allowed? out of scope
S_n, S_x orbit $S_x \circ f \circ S_n$	How many ways can you place n plain balls into x plain boxes, with no other rules on placement? out of scope	How many ways can you place n plain balls into x plain boxes, with no multi-packs allowed? out of scope	How many ways can you place n plain balls into x plain boxes, with no empty boxes allowed? out of scope

Application of bijection: cardinality of infinite set

Not in exam

$$|\mathcal{S}| \begin{cases} = 0 & \text{if } \mathcal{S} = \varnothing \\ < \infty & \text{if } \mathcal{S} \text{ is finite} \end{cases}$$

- **Definition** $\mathcal{S}$ is countable if either $\mathcal{S}$ is finite, or $\mathcal{S}$ is infinite with an *bijection*[2] with $\mathbb{N}$
- $|\mathbb{N}| := \aleph_0$ (Aleph null), it means "countably infinite" think of it as ∞
- A set $\mathcal{S}$ is countable if $|\mathcal{S}| \leq \aleph_0$, a set is uncountable if $|\mathcal{S}| > \aleph_0$
- Cantor's diagonal argument for bijection of infinite set
  - Let $\mathbb{N}$ be the set of natural number
  - Let $\mathbb{E}$ be the set of positive even integers
  - Which set has more elements?

[2] one-to-one (injective) onto (surjective)

Algebraic methods in enumerative combinatorics

- Question: is there a more systemic way to counting?
  Answer: yes

- Recall the very beginning:

	single term	double term	triple term	quadruple term
$\lvert A\rvert$	1			
$\lvert A\cup B\rvert$	2	1		
$\lvert A\cup B\cup C\rvert$	3	3	1	
$\lvert A\cup B\cup C\cup D\rvert$	4	6	4	1

	power-1 term	power-2 term	power-3 term	power-4 term
$(1+x)^1$	1			
$(1+x)^2$	2	1		
$(1+x)^3$	3	3	1	
$(1+x)^4$	4	6	4	1

  we can count things using the coefficients of polynomials ($\because$ they have the same algebra)

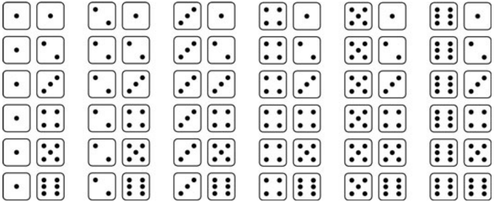
- We can count things using the coefficients of polynomials because they have a **bijection**.

Counting via polynomial coefficient, example

- **Example** For a, b, c in $\mathbb{Z}_+$, if $2 \le a \le 5, 3 \le b \le 6$ and $4 \le c \le 7$, how many solutions to $a + b + c = 17$?
  - It is the coefficient of x^{17} in the polynomial
  $$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$
- **Example** You distribute 8 identical cookies among 3 children. Each child can receive at least 2 cookies and no more than 4 cookies. How many ways can you distribute?
  - It is the coefficient of x^8 in the polynomial
  $$(x^2 + x^3 + x^4)(x^2 + x^3 + x^4)(x^2 + x^3 + x^4).$$
- How come? Because there is a bijection between the problem and the polynomial coefficient.

Six-sided die

- Die = singular form of dice in old-fashioned English
- All the possible outcome of 2d6 (toss a six-sided die twice)



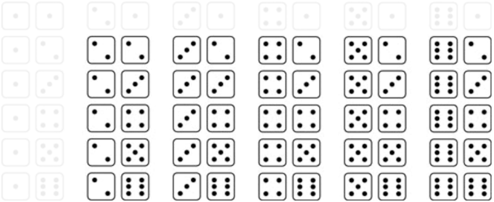
- Consider a polynomial $x^1 + x^2 + x^3 + x^4 + x^5 + x^6$
  All the terms in $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$

$$\begin{array}{cccccc} x^1x^1 & x^2x^1 & x^3x^1 & x^4x^1 & x^5x^1 & x^6x^1 \\ x^1x^2 & x^2x^2 & x^3x^2 & x^4x^2 & x^5x^2 & x^6x^2 \\ x^1x^3 & x^2x^3 & x^3x^3 & x^4x^3 & x^5x^3 & x^6x^3 \\ x^1x^4 & x^2x^4 & x^3x^4 & x^4x^4 & x^5x^4 & x^6x^4 \\ x^1x^5 & x^2x^5 & x^3x^5 & x^4x^5 & x^5x^5 & x^6x^5 \\ x^1x^6 & x^2x^6 & x^3x^6 & x^4x^6 & x^5x^6 & x^6x^6 \end{array} \Longrightarrow \begin{array}{cccccc} x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ x^4 & x^5 & x^6 & x^7 & x^8 & x^9 \\ x^5 & x^6 & x^7 & x^8 & x^9 & x^{10} \\ x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11} \\ x^7 & x^8 & x^9 & x^{10} & x^{11} & x^{12} \end{array}$$

$$1x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12}$$

Defective six-sided die

- Suppose the die is defective that output 1 is impossible. All the possible outcome of 2d6



- Consider a polynomial $\color{red}{0}x^1 + x^2 + x^3 + x^4 + x^5 + x^6$
  All the terms in $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$, where we ignore x^1

$$\begin{array}{cccccc} x^1x^1 & x^2x^1 & x^3x^1 & x^4x^1 & x^5x^1 & x^6x^1 \\ x^1x^2 & x^2x^2 & x^3x^2 & x^4x^2 & x^5x^2 & x^6x^2 \\ x^1x^3 & x^2x^3 & x^3x^3 & x^4x^3 & x^5x^3 & x^6x^3 \\ x^1x^4 & x^2x^4 & x^3x^4 & x^4x^4 & x^5x^4 & x^6x^4 \\ x^1x^5 & x^2x^5 & x^3x^5 & x^4x^5 & x^5x^5 & x^6x^5 \\ x^1x^6 & x^2x^6 & x^3x^6 & x^4x^6 & x^5x^6 & x^6x^6 \end{array} \Longrightarrow \begin{array}{cccccc} x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ x^4 & x^5 & x^6 & x^7 & x^8 & x^9 \\ x^5 & x^6 & x^7 & x^8 & x^9 & x^{10} \\ x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11} \\ x^7 & x^8 & x^9 & x^{10} & x^{11} & x^{12} \end{array}$$

$$1x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12}$$

Distribute cookies to two children

- How many ways to distribute 8 cookies to two children with at least 1 cookies and no more than 6 cookies

$$\begin{array}{cccccc} x^1x^1 & x^2x^1 & x^3x^1 & x^4x^1 & x^5x^1 & x^6x^1 \\ x^1x^2 & x^2x^2 & x^3x^2 & x^4x^2 & x^5x^2 & x^6x^2 \\ x^1x^3 & x^2x^3 & x^3x^3 & x^4x^3 & x^5x^3 & x^6x^3 \\ x^1x^4 & x^2x^4 & x^3x^4 & x^4x^4 & x^5x^4 & x^6x^4 \\ x^1x^5 & x^2x^5 & x^3x^5 & x^4x^5 & x^5x^5 & x^6x^5 \\ x^1x^6 & x^2x^6 & x^3x^6 & x^4x^6 & x^5x^6 & x^6x^6 \end{array} \Longrightarrow \begin{array}{cccccc} x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ x^4 & x^5 & x^6 & x^7 & x^8 & x^9 \\ x^5 & x^6 & x^7 & x^8 & x^9 & x^{10} \\ x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11} \\ x^7 & x^8 & x^9 & x^{10} & x^{11} & x^{12} \end{array}$$

- How many ways to distribute 7 cookies to two children with at least 2 cookies and no more than 5 cookies

$$\begin{array}{cccccc} x^1x^1 & x^2x^1 & x^3x^1 & x^4x^1 & x^5x^1 & x^6x^1 \\ x^1x^2 & x^2x^2 & x^3x^2 & x^4x^2 & x^5x^2 & x^6x^2 \\ x^1x^3 & x^2x^3 & x^3x^3 & x^4x^3 & x^5x^3 & x^6x^3 \\ x^1x^4 & x^2x^4 & x^3x^4 & x^4x^4 & x^5x^4 & x^6x^4 \\ x^1x^5 & x^2x^5 & x^3x^5 & x^4x^5 & x^5x^5 & x^6x^5 \\ x^1x^6 & x^2x^6 & x^3x^6 & x^4x^6 & x^5x^6 & x^6x^6 \end{array} \Longrightarrow \begin{array}{cccccc} x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ x^4 & x^5 & x^6 & x^7 & x^8 & x^9 \\ x^5 & x^6 & x^7 & x^8 & x^9 & x^{10} \\ x^6 & x^7 & x^8 & x^9 & x^{10} & x^{11} \\ x^7 & x^8 & x^9 & x^{10} & x^{11} & x^{12} \end{array}$$

$$\Longleftrightarrow (x^2+x^3+x^4+x^5)(x^2+x^3+x^4+x^5)$$

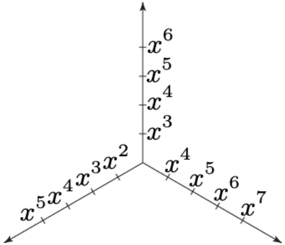
Number of integer solutions to an equation

- **Example** How many solution to the equation $a+b+c=17$ if a,b,c are integers and $2 \le a \le 5, 3 \le b \le 6$ and $4 \le c \le 7$?
  - The solution is the coefficient of x^{17} in the polynomial

$$(x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(x^4+x^5+x^6+x^7)$$

(which is 3)

  - The same as plotting all the (i,j,k) coordinate of the outcomes and find the total number of terms that has power 17.



Power series $G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$

- **Example** Consider a 1-set $\{e_1\}$

  1 way to select zero elements

  1 way to select one element

  0 way to select more than 1 element

  $$1x^{\boxed{0}} + 1x^{\boxed{1}} + 0x^2 + 0x^3 + \dots = 1 + x$$

- **Example** Consider a 2-set $\{e_1, e_2\}$

  1 way to select zero elements

  2 way to select one element

  1 way to select two elements (we take combination)

  0 way to select more than 2 elements

  $$1x^0 + 2x^1 + x^2 + 0x^3 + \dots = 1 + 2x + x^2$$

- $1 + 2x + x^2 = (1 + x) \cdot (1 + x)$

  $$\underbrace{(1+x)}_{\text{select from } \{e_1\}} \underbrace{\cdot}_{\text{product rule}} \underbrace{(1+x)}_{\text{select from } \{e_2\}} = \underbrace{(1+x)^2}_{\text{select from } \{e_1, e_2\}} = 1 + 2x + x^2$$

- In fact these are binomial expansion of $(1 + x)^k$

- What about permutation: $P(n, k) = C(n, k) \cdot k!$

Sequence and generating function

- **Example** Given a sequence $\underbrace{\{1,1,1,1,1,1\}}_{six\ 1}$, its generating function is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

  Note that we have three representations of the same thing

$$\underbrace{f_n = \begin{cases} 1 & n \in \{0,1,2,3,4,5\} \\ 0 & \text{other wise} \end{cases}}_{\text{definition form}} \quad \Longleftrightarrow \quad \underbrace{\{1,1,1,1,1,1\}}_{\text{a list}} \quad \Longleftrightarrow \quad \underbrace{1 + x + x^2 + x^3 + x^4 + x^5}_{\text{generating function}}$$

- **Example** For sequence $\{0,1,2,3\}$, its generating function is $x + 2x^2 + 3x^3$.

- **Example** For sequence $\{0,1,0,1,0,1,0,1,...\}$, its generating function is $x + x^3 + x^5 + x^7 + ...$

Example of product rule and generating function

- **Example** A word must be started with a prefix, followed by a vowels, and ends in a suffix. How many words can you build from
$$\begin{aligned} \texttt{prefixes} &= \{\texttt{qu}, \texttt{s}, \texttt{t}\} \\ \texttt{vowels} &= \{\texttt{a}, \texttt{i}, \texttt{oi}\} \\ \texttt{suffixes} &= \{\texttt{d}, \texttt{ff}, \texttt{ck}\} \end{aligned}$$
?

- Solution by enumerating the $3^3 = 27$ outcomes

```
sad  sid  tad  tid
quad  quid  sack  saff  sick  siff  soid  tack  taff  tick  tiff  toid
quack  quaff  quick  quiff  quoid  soick  soiff  toick  toiff  quoick  quoiff
```

There are 4 length-3 words, 12 length-4 words, 9 length-5 words, and 2 length-6 words.

- How to count number of length-5 words without enumeration? We construct a product of generating function
$$\begin{aligned} (\texttt{qu} + \texttt{s} + \texttt{t})(\texttt{a} + \texttt{i} + \texttt{oi})(\texttt{d} + \texttt{ff} + \texttt{ck}) &= (xx + x + x)(x + x + xx)(x + xx + xx) \\ &= (2x + x^2)(2x + x^2)(x + 2x^2) \\ &= 4x^3 + 12x^4 + 9x^5 + 2x^6. \end{aligned}$$

(we don't care about the exact letter, we replace each letter by x)
(Why qu is xx: the xx represents "select or not select 2 letters")

- Coefficient of $x^5 = \#$ of length-5 words $= 9$.

Product rule and sum rule, in generating function

- Let $\begin{cases} A(x) \text{ be the gen. func. for selecting items from set } \mathcal{A} \\ B(x) \text{ be the gen. func. for selecting items from set } \mathcal{B} \end{cases}$

- **Example** Rolling a six-sided die four times. By product rule, the GF is

$$C(x) = (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^4$$

  The number of ways that the result add up to 20 is the coefficient of x^{20}.
  Then how do you find the coefficient: multinomial coefficient.

- **Product rule (GF ver.)**
  The GF for the set $\mathcal{C} := A \times B$ is $C(x) = A(x)B(x)$.

- **Sum rule of disjoint set (GF ver.)**
  If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, the GF for the set $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is $C(x) = A(x) + B(x)$

- Is there a GF for inclusion-exclusion principle?
  Yes but too advanced for this course.

Coefficient in infinite series

On $1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$, take differentiation $n-1$ times gives

$$\sum_{k=n-1}^{\infty} k(k-1)(k-2)\cdots(k-n+2)x^{k-n+1} = \frac{(n-1)!}{(1-x)^n}$$

Let $r = k - n + 1$ then $k = r + n - 1$

$$\begin{aligned}
\sum_{r+n-1=n-1}^{\infty} (r+n-1)(r+n-2)(r+n-3)\cdots(r+1)x^r &= \frac{(n-1)!}{(1-x)^n} \\
\iff \quad \sum_{r=0}^{\infty}(r+n-1)(r+n-2)(r+n-3)\cdots(r+1)\frac{r!}{r!}x^r &= \frac{(n-1)!}{(1-x)^n} \\
\iff \quad \sum_{r=0}^{\infty} \frac{(r+n-1)!}{r!}x^r &= \frac{(n-1)!}{(1-x)^n} \\
\iff \quad \sum_{r=0}^{\infty} \frac{(r+n-1)!}{(n-1)!r!}x^r &= \frac{1}{(1-x)^n} = (1+x+x^2+\cdots)^n
\end{aligned}$$

$$\boxed{(1+x+x^2+\cdots)^n = \sum_{r=0}^{\infty}\binom{r+n-1}{r}x^r}$$

- **Example** $[x^{22}](1+x+x^2+\cdots)^1 = \binom{22+1-1}{22} = 1, \quad [x^{22}](1+x+x^2+\cdots)^8 = \binom{22+8-1}{22} = \binom{29}{22} = \frac{29!}{22!7!}$

Example of coefficient in infinite series

$$\boxed{(1+x+x^2+\cdots)^n \;=\; \sum_{r=0}^{\infty}\binom{r+n-1}{r}x^r}$$

Example $[x^{22}](x+x^2+\cdots)^3$ is the same as $[x^{19}](1+x^2+\cdots)^3$, which is $\binom{19+3-1}{19}=\binom{21}{19}=210.$

Or,
$$\begin{aligned}
(x+x^2+\cdots)^3 &= (-1+1+x+x^2+\cdots)^3\\
&= \Big(-1+(1+x+x^2+\cdots)\Big)^3\\
&= \sum_{k=0}^{3}\binom{3}{k}(-1)^{3-k}(1+x+x^2+\cdots)^k\\
&= \binom{3}{0}(-1)^{3-0}(1+x+x^2+\cdots)^0+\binom{3}{1}(-1)^{3-1}(1+x+x^2+\cdots)^1\\
&\quad+\binom{3}{2}(-1)^{3-2}(1+x+x^2+\cdots)^2+\binom{3}{3}(-1)^{3-3}(1+x+x^2+\cdots)^3\\
&= -1+3(1+x+x^2+\cdots)-3(1+x+x^2+\cdots)^2+(1+x+x^2+\cdots)^3
\end{aligned}$$

The coefficient is thus

$$0+3\binom{22+1-1}{22}-3\binom{22+2-1}{22}+\binom{22+3-1}{22} \;=\; 3-3(23)+276=210.$$

Using generating function to solve difficult counting problem ... (1/2)

- **Example** How many ways to fill a bag with n fruits if $\begin{cases} \text{number of apples must be even} \\ \text{number of bananas must be a multiple of 5} \\ \text{at most four oranges} \\ \text{at most one pear} \end{cases}$

- Solve it using generating function
  - We can select apples in way of $\{1, 0, 1, 0, \dots\}$ $\begin{cases} \text{1 way for zero apple} \\ \text{0 way for one apple} \\ \text{1 way for two apples} \\ \text{0 way for three apples} \\ \text{so on} \end{cases}$

    The generating function for apple is $A(x) = 1 + x^2 + x^4 + x^6 + \dots$
  - The generating function for Banana, $B(x) = 1 + x^5 + x^{10} + x^{15} + \dots$
  - The generating function for orange, $O(x) = 1 + x^1 + x^2 + x^3 + x^4$
  - The generating function for pear, $P(x) = 1 + x^1$

- By product rule (GF ver.), the generating function of slecting the fruits is

$$A(x)B(x)O(x)P(x) \;=\; \Big(1 + x^2 + x^4 + x^6 + \dots\Big)\Big(1 + x^5 + x^{10} + x^{15} + \dots\Big)\Big(1 + x^1 + x^2 + x^3 + x^4\Big)\Big(1 + x^1\Big)$$

  Let's call $A(x)B(x)O(x)P(x)$ as $\texttt{Fruit}(x)$.

Using generating function to solve difficult counting problem ... (2/2)

$$
\begin{aligned}
\texttt{Fruit}(x) &= \left(1+x^2+x^4+\dots\right)\left(1+x^5+x^{10}+\dots\right)\left(1+x^1+x^2+x^3+x^4\right)\left(1+x^1\right) \\
&= \left(1+x^2+x^4+\dots\right)\left(1+x^5+x^{10}+\dots\right)\left(\frac{1-x^5}{1-x}\right)\left(1+x^1\right) && \text{geometric sum} \\
&= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)\left(1+x^1\right) && \text{geometric series} \\
&= \frac{1}{(1-x)\cancel{(1+x^1)}}\frac{1}{\cancel{1-x^5}}\frac{\cancel{1-x^5}}{1-x}\cancel{(1+x^1)} \\
&= \frac{1}{(1-x)^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} && \text{differentiation} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(1+x+x^2+x^3+\dots\right) && \text{geometric series} \\
&= 1+2x^1+3x^2+4x^3+\dots \\
&= (n+1)x^n\Big|_{n=0}+(n+1)x^n\Big|_{n=1}+(n+1)x^n\Big|_{n=2}+(n+1)x^n\Big|_{n=3}+\cdots
\end{aligned}
$$

So there are $(n+1)$ ways to select n fruits.

Multiplicative inverse of power series

- **Definition** A polynomial $B(x)$ is a inverse of $A(x)$ if $B(x)A(x) = 1$.

- **Example** $1 - x$ is the inverse of $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$

$$(1-x)\sum_{k=0}^{\infty} x^k \;=\; (1-x)\big(1 + x + x^2 + x^3 + \cdots\big) \;=\; \begin{array}{r} 1 + x + x^2 + x^3 + \cdots \\ -x - x^2 - x^3 - \cdots \end{array} \;=\; 1.$$

  In fact, by geometric series, $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.

- **Generalization** By geometric sum, we have

$$1 + rx + r^2x^2 + \cdots = \frac{1}{1-rx}$$

  Illustration using multiplicative inverse

$$(1-rx)\big(1 + rx + r^2x^2 + r^3x^3 + \cdots\big) \;=\; \begin{array}{r} 1 + rx + r^2x^2 + r^3x^3 + \cdots \\ -rx - r^2x^2 - r^3x^3 - \cdots \end{array} \;=\; 1.$$

What's the big deal of generating function? Not in exam

- A more systematic approach to counting: you count by using polynomial coefficient.
- However, you need a lots of mathematics to use generating function
  - Finite sum and infinite sum
  - Differentiation and integration
  - Partial fraction
- Generating function can be used to solve recursion problem
  - Such as finding the closed-form solution of Fibonacci number as $f_n = \frac{(-1)^{n+1}}{\sqrt{5}}\left(\varphi_-^{n+1} - \varphi_+^{n+1}\right)$, where $\varphi_\pm = \frac{-1 \pm \sqrt{5}}{2}$ is known as golden ratio.
- In fact generating function is called *z-transform* in digital signal processing.
  - It is called z because it uses complex number.
  - In fact, given a generating function, we can get back the series via inverse Z-transform via
  $$x[n] = \frac{1}{2\pi\sqrt{-1}} \oint_C X(z) z^{n-1} \mathrm{d}z,$$
  in which you need the knowledge of complex number and complex integration.

Partial fraction

- **Example** $\dfrac{1}{(1-x)(1-2x)} = \dfrac{?}{1-x} + \dfrac{?}{1-2x}$

$$\frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} \iff 1 = A(1-2x) + B(1-x).$$

Put $x = 1$ gives $A = -1$. Put $x = \frac{1}{2}$ gives $B = 2$, so $\dfrac{1}{(1-x)(1-2x)} = \dfrac{-1}{1-x} + \dfrac{2}{1-2x}$.

- **Example** $\dfrac{17x-53}{x^2-2x+15} = \dfrac{?}{x-5} + \dfrac{?}{x+3}$

$$\frac{17x-53}{x^2-2x+15} = \frac{17x-53}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \iff 17x-53 = A(x+3) + B(x-5).$$

Put $x = 5$ gives $A = 4$ and put $x = -3$ gives $B = 13$. Hence $\dfrac{17x-53}{x^2-2x+15} = \dfrac{4}{x-5} + \dfrac{13}{x+3}$

- **Example** $\dfrac{10+35}{(x+4)^2} = \dfrac{10}{x+4} + \dfrac{-5}{(x+4)^2}$

$$\frac{10x+35}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2} \iff 10x+35 = A(x+4) + B.$$

Put $x = -4$ gives $B = -5$ and put $x = 0$ with $B = -5$ gives $A = 10$.

Solving recursion via generating function, example 1 ... (1/2)

- Consider a function defined recursively as $f_n = \begin{cases} 2 & n = 0 \\ 3f_{n-1} - 1 & n \geq 1 \end{cases}$. What is the general formula of f_n?
- Enumerating the solution for multiple n and then guessing the solution is not very helpful.
- Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. Now $f_n = 3f_{n-1} - 1 \iff f_n - 3f_{n-1} = -1$.

$$\begin{array}{rcll} F(x) & = & f_0 + f_1x^1 + f_2x^2 + f_3x^3 + \cdots & \text{by definition} \\ -3xF(x) & = & -3f_0x^1 - 3f_1x^2 - 3f_2x^3 - \cdots & \text{multiply } -3x \\ (1-3x)F(x) & = & f_0 + (f_1 - 3f_0)x^1 + (f_2 - 3f_1)x^2 + (f_3 - 3f_2)x^3 + \cdots & \text{add the two equations} \\ & = & 2 - x^1 - x^2 - x^3 - \cdots & \text{by} \\ & = & 2 + (1-1) - x^1 - x^2 - x^3 - \cdots & \\ & = & 3 - (1 + x^1 + x^2 + x^3 + \cdots) & \\ & = & 3 - \dfrac{1}{1-x} & \text{geometric series} \end{array}$$

- Now we have

$$(1-3x)F(x) = 3 - \frac{1}{1-x} \iff F(x) = \frac{3}{1-3x} - \frac{1}{(1-x)(1-3x)}$$

Solving recursion via generating function, example 1 ... (2/2)

- Apply partial fraction

$$\frac{1}{(1-x)(1-3x)} = \frac{\frac{1}{1-3}}{1-x} + \frac{\frac{1}{1\frac{1}{3}}}{1-3x} = \frac{-\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}$$

- Then $(1-3x)F(x) = 3 - \dfrac{1}{1-x}$ gives

$$\begin{aligned} F(x) &= \frac{3}{1-3x} + \frac{\frac{1}{2}}{1-x} - \frac{\frac{3}{2}}{1-3x} \\ &= \frac{\frac{3}{2}}{1-3x} + \frac{\frac{1}{2}}{1-x} \\ &= \frac{3}{2}\frac{1}{1-3x} + \frac{1}{2}\frac{1}{1-x} \\ &= \frac{3}{2}\Big(1 + 3x + (3x)^2 + (3x)^3 + \cdots\Big) + \frac{1}{2}\Big(1 + x + x^2 + x^3 + \cdots\Big) \end{aligned}$$

- Therefore

$$f_n = \frac{3}{2}\cdot 3^n + \frac{1}{2}\cdot 1^n = \frac{3}{2}\cdot 3^n + \frac{1}{2}$$

  You can try with $n = \{0, 1, 2, 3\}$

Solving recursion via generating function, example 2

- Find the closed-form expression of the recursion $f_n = \begin{cases} b & n = 0 \\ f_{n-1} + c & n \geq 1 \end{cases}$

- Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. Now $f_n = f_{n-1} + c \iff f_n - f_{n-1} = c$.

$$
\begin{aligned}
F(x) &= f_0 + f_1 x^1 + f_2 x^2 + \cdots && \text{by definition} \\
-\tfrac{1}{x} F(x) &= -f_0 \tfrac{1}{x} - f_1 - f_2 x^1 - f_3 x^2 - \cdots && \text{multiply } -\tfrac{1}{x} \\
(1 - \tfrac{1}{x}) F(x) &= -f_0 \tfrac{1}{x} + (f_0 - f_1) x^1 + (f_1 - f_2) x^2 + (f_2 - f_3) x^3 + \cdots && \text{add the two equations} \\
&= -\tfrac{b}{x} - c x^1 - c x^2 - c x^3 - \cdots && \text{by and } f_0 = b \\
(1 - x) F(x) &= b + c(x^1 + x^2 + x^3 + \cdots) && \text{multiply } -x \\
&= b - c + c(1 + x^1 + x^2 + x^3 + \cdots) && \\
&= b - c + c \frac{1}{1 - x} && \text{geometric series} \\
F(x) &= \frac{b - c}{1 - x} + c \frac{1}{(1 - x)^2} && \text{divide by} (1 - x) \\
&= (b - c) \frac{1}{1 - x} + c \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1 - x} && \frac{1}{(1 - x)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1 - x} \\
&= (b - c)(1 + x + x^2 + \ldots) + c \frac{\mathrm{d}}{\mathrm{d}x} (1 + x + x^2 + \ldots) && \text{geometric series} \\
&= (b - c)(1 + x + x^2 + \ldots) + c(1 + 2x + 3x^2 + \ldots) && \\
&= b + (b + c)x + (b + 2c)x^2 + (b + 3c)x^3 + \ldots && \\
f_n &= b + nc &&
\end{aligned}
$$

Solving recursion via generating function, example 3 ... (1/2)

Find the closed-form expression of the recursion $f_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ f_{n-2} + 1 & n \geq 2 \end{cases} \iff \boxed{f_n - f_{n-2} = 1}$

$$
\begin{aligned}
F(x) &= f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \cdots && \text{by definition} \\
-x^2 F(x) &= \qquad -f_0 x^2 - f_1 x^3 - \cdots && \text{multiply } -x^2 \\
(1 - x^2) F(x) &= f_0 + f_1 x + (f_2 - f_0) x^2 + (f_3 - f_1) x^3 + \cdots && \text{add the two equations} \\
&= x - x^2 - x^3 - \cdots && \text{by } \boxed{} \text{ and } f_0 = 0,\ f_1 = 1 \\
&= 1 - 1 - x - x^2 - \cdots \\
(1 + x)(1 - x) F(x) &= 1 - \frac{1}{1 - x} && \text{geometric series} \\
F(x) &= \frac{1}{(1 - x)(1 + x)} + \frac{-1}{(1 + x)(1 - x)^2} && \text{divide by} (1 + x)(1 - x) \\
&= \frac{\frac{1}{2}}{1 - x} + \frac{\frac{1}{2}}{1 + x} + \frac{\frac{-1}{4}}{1 + x} + \frac{\frac{-1}{4}}{1 - x} + \frac{\frac{1}{2}}{(1 - x)^2} && \text{partial fraction} \\
&= \frac{1}{4} \frac{1}{1 - x} + \frac{1}{4} \frac{1}{1 - (-1)x} + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1 - x} \\
&= \tfrac{1}{4}\left(1 + x + x^2 + \cdots\right) + \tfrac{1}{4}\left(1 - x + x^2 - \cdots\right) + \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + x^2 + \cdots\right) \\
&= \tfrac{1}{4}\left(1 + x + x^2 + \cdots\right) + \tfrac{1}{4}\left(1 - x + x^2 - \cdots\right) + \tfrac{1}{2}\left(1 + 2x + 3x^2 + \cdots\right) \\
&= \left(\tfrac{1}{4} + \tfrac{-1}{4} + \tfrac{1}{2}\right) + \left(\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{2}{2}\right) x + \left(\tfrac{1}{4} + \tfrac{-1}{4} + \tfrac{3}{2}\right) x^2 + \cdots \\
&= \tfrac{1 - (-1)^0 + 2(1)}{4} + \tfrac{1 - (-1)^1 + 2(2)}{4} x + \tfrac{1 - (-1)^2 + 2(3)}{4} x^2 + \cdots \\
f_n &= \tfrac{1 - (-1)^n + 2n}{4}
\end{aligned}
$$

Double check

- $f_0 = \frac{1}{4}\left(1 - (-1)^0 + 0\right) = 0$ true, $\quad f_1 = \frac{1}{4}\left(1 - (-1)^1 + 2(1)\right) = 1$ true
- $f_2 = \frac{1}{4}\left(1 - (-1)^2 + 2(2)\right) = 1 = f_0 + 1$ true, $\quad f_3 = \frac{1}{4}\left(1 - (-1)^3 + 2(3)\right) = 2 = f_1 + 1$ true

Solving recursion via generating function, example 4

- Find the closed-form expression of the recursion $f_n = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 3f_{n-1} - 2f_{n-2} & n \geq 2 \end{cases}$
- Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. By $f_n = 3f_{n-1} - 2f_{n-2} \iff f_n - 3f_{n-1} + 2f_{n-2} = 0$, we have

$$\begin{aligned}
F(x) &= f_0 + f_1 x^1 + \quad f_2 x^2 + \quad f_3 x^3 + \cdots \\
-3xF(x) &= \quad -3f_0 x^1 - 3f_1 x^2 - 3f_2 x^3 - \cdots \\
2x^2 F(x) &= \qquad\qquad +2f_0 x^2 + 2f_1 x^3 + \cdots \\
(1 - 3x + 2x^2)F(x) &= f_0 + (f_1 - 3f_0)x + (f_2 - 3f_1 + 2f_0)x^2 + (f_3 - 3f_2 + 2f_1)x^3 + \cdots \\
&= 1 + (3 - 3)x + 0x^2 + 0x^3 + \cdots \\
&= 1 \\
F(x) &= \frac{1}{2x^2 - 3x + 1} \\
&= \frac{1}{(1 - 2x)(1 - x)} \\
&= \frac{2}{1 - 2x} + \frac{-1}{1 - x} \\
&= 2\Big(1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots\Big) - \Big(1 + x + x^2 + \cdots\Big) \\
f_n &= 2^{n+1} - 1
\end{aligned}$$

Verify with $f_n = 1, 3, 7, 15, 31, 63, \ldots$

Techniques in generating function

- Steps
  - Let $F(x) = f_0 + f_1x^1 + f_2x^2 + f_3x^3 + \cdots$
  - Based on the recursion definition, multiply $F(x)$ by a polynomial in x
  - Work on the algebra to get a power series
- Familiar with common power series
  - $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{1-r^n}{1-r}$
  - $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$
  - $1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{1-x}$
- Familiar with partial fraction tricks, such has

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{x-a} + \frac{\frac{1}{b-a}}{x-b}$$

Chapter 7 in Discrete Mathematics and Its application →

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$ $= 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^kx^k$ $= 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^nx^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$ $= 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\dfrac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\dfrac{1}{1-ax} = \sum_{k=0}^{\infty} a^kx^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\dfrac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k)$ $= C(n+k-1,n-1)$
$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^kx^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^kC(n+k-1,k)$ $= (-1)^kC(n+k-1,n-1)$
$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^kx^k$ $= 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k$ $= C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$

Advanced generating function

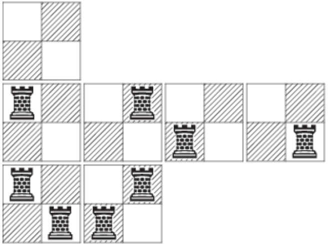
Not in exam

- Fibonacci sequence $f_n = \{0, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$

$$f_n = \frac{(-1)^{n+1}}{\sqrt{5}}\left(\varphi_-^{n+1} - \varphi_+^{n+1}\right), \qquad \varphi_\pm = \frac{-1 \pm \sqrt{5}}{2} \text{ golden ratio}$$

- Rook polynomial $\Longleftrightarrow$ ways of nonattacking rooks in chessboard

$$R_{2,2}(x) = 1 + 4x + 2x^2$$

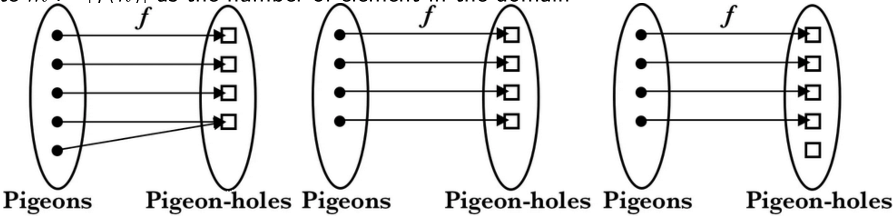


no rook: 1 way
1 rook: 4 ways
2 rooks: 2 ways

- What is the number of ways to place 8 rooks in a 8-by-8 chess board that none of the rook are attacking?
- What about bishop? Rotate the chessboard 45° gives you a rook!
- Queen = Rook + bishop.

Pigeonhole principle and function mapping

- Suppose we are mapping n elements from the domain to a range via a function f
  - Denote $|n|$ as the number of element in the domain
  - Denote $m := |f(n)|$ as the number of element in the domain



- If $|n| > |f(n)|$, then the function is a non-injective surjective function (not a bijection)
- If $|n| = |f(n)|$, then the function is a injective surjective function (bijection)
- If $|n| < |f(n)|$, then the function is an injective non-surjective function (not a bijection).
- **Pigeonhole principle**
  - If you put n pigeons into m pigeonholes and $n > m$, then at least 1 pigeonhole must contains more than 1 pigeons. To be exact, that pigeonhole contains $\lfloor (n-1)/m \rfloor + 1$ pigeons.
  - If you put $kn + 1$ pigeons into n pigeonholes and $k \in \mathbb{Z}_+$, then at least 1 pigeonhole must contain more than $k + 1$ pigeons.

Pigeonhole principle example .. (1/2)

- **Example** Put five pigeons into two pigeonholes A, B, then one of the pigeonhole will has at least 3 pigeons.
  - By enumeration:

$AAAAA$	A has $5 > 3$ pigeons
$AAAAB$	A has $4 > 3$ pigeons
$AAABB$	A has $3 = 3$ pigeons
$AABBB$	B has $3 = 3$ pigeons
$ABBBB$	B has $4 > 3$ pigeons
$BBBBB$	B has $5 > 3$ pigeons

- **Example** In a group of 13 people, we have 2 or more who are born in the same month.
  - Number of pigeons: 13 (number of people)
  - Number of pigeonholes: 12 (number of months)
- **Example** In a group of $n > 1$ people, each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.
  - Number of pigeons: n (number of people)
  - Number of pigeonholes: $n - 1$ (range of number of handshakes)
- **Example** For any choice of 10 numbers in $\{1, 2, \ldots, 19\}$, there are two that add up to 19.
  - we partition $\{1, 2, \ldots, 19\}$ into 9 subsets: $\{1, 18\}, \{2, 17\}, \{3, 16\}, ..., \{9, 10\}$.
  - we are picking 10 numbers, by the pigeonhole principle, there must be two chosen numbers lie in the same bracket
- **Example** Suppose there are 26 students and 7 cars to transport them. Then at least 1 car have 4 or more passengers. $\lfloor(26 - 1)/7\rfloor + 1 = 3 + 1 = 4$.

Pigeonhole principle example .. (2/2)

- **Example** A class of 20 students got their exam scores, I told them the class average for the test was 8 out of a max of 10 . Then someone in the class must have scored at least an 8/10.
  - **Proof by contradiction** Let x_i be the scores of the student.
  - For contradiction, assume the statement is false. I.e., nobody get more than $8/10$. This means $x_i < 8$ for all i.
  - The average of all 20 students is then

$$\text{class average } = \frac{x_1 + x_2 + \cdots + x_{20}}{20} \overset{\text{assumption}}{<} \frac{8+8+\cdots+8}{20} = \frac{160}{20} = 8.$$

  This contradicts with , therefore assumption is false, and someone get more than $8/10$.

- **Example** For any 5 points in the unit square $[0,1] \times [0,1]$, there are two points that are within $\frac{1}{\sqrt{2}}$ distance of each other
  - We partition $[0,1] \times [0,1]$ into 4 quadrants.
  - By the pigeonhole principle, two of the points are in the same quadrant.
  - The farthest two points can be in the same quadrant is the length of the diagonal of the square, which is $\frac{1}{\sqrt{2}}$.

Now you see pigeonhole principle is *existential*: it only gives the information of "there exists in an object that has property X", it does not specify which object has the property X.

The relationship of 6 people

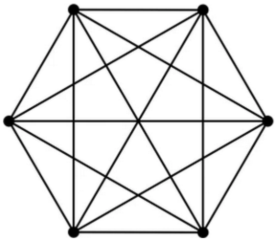
Not in exam

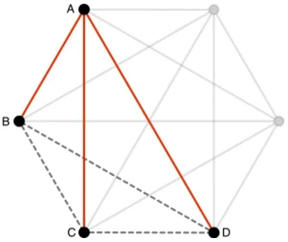
- **Theorem** Suppose there are 6 people $\{1, 2, 3, 4, 5, 6\}$, and for each pair (i, j) there is a relationship of $\{\texttt{friends}, \texttt{stranger}\}$. Then there will be at least 3 people that they are friends of each other or they are strangers.

- The above two situations **always** hold no matter how you pick 6 people in this world. This statement does not hold if you consider less than 6 people.

- Ramsey number $R(3, 3) = 6$

- Proved by pigeonhole principle on a complete graph: 5 edges, 2 group, 3 connections
  - Claim: for A, at least 3 edges on friends / stranger
  - Pigeon: 3
  - Pigeonhole: $\lfloor \frac{5}{2} \rfloor$





What's the big deal of pigeonhole principle? Not in exam

- Pigeonhole principle, founded by Peter Gustav Lejeune Dirichlet, gives us a guarantee on what can happen in the "worst case scenario".
- Pigeonhole principle is a special case of Ramsey's theory.
- Ramsey theory: "how big must some structure be to guarantee a particular property holds?"
  - **"Complete disorder is impossible" - Theodore S. Motzkin**
- Frank Ramsey, who proposed the Ramsey's theory in 1928 (at age 25), is a friend of Ludwig Wittgenstein (who invented the *truth table*). Wittgenstein is PhD a student of Bertrand Russell (logic).

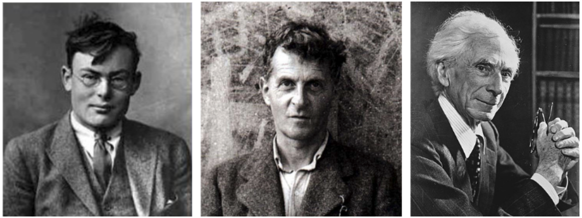


Figure: Ramsey, Wittgenstein and Russell

- Ramsey theory is an entire branch of mathematics that is too difficult (at least for me)

Pigeonhole principle in computer science

Not in exam

- Pigeonhole principle is not a theorem but rather a proof-technique.
- You will not see the power of pigeonhole principle in this course. You will only experience its power in later course.
- **Example** All *list* with more than n^2 distinct numbers has a monotone sublist of length greater than n.
  - e.g. $n = 3$, so we need to build a list of at least $n^2 = 9$ distinct numbers.
  - Say we randomly generate $10 > n^2 = 9$ distinct integers as

    $$\texttt{list} = \{1, 10, 2, 9, 3, 8, 4, 7, 5, 6\}$$

    A sublist of `list`, in which you can skip elements, is

    $$\texttt{sublist} = \{1, 2, 3, 4, 5, 6\}, \quad |\texttt{sublist}| = 6 > 3 =: n.$$
- **Example** Any *comparison-based sorting algorithm* must perform $\Omega(n \log n)$ comparisons to sort n elements in the worst case.
  - Proof by contradiction: if the algorithm makes fewer comparisons than this, by pigeonhole principle, there must be some pair of inputs that the algorithm wouldn't be able to distinguish, since there are more possible inputs than configurations of the algorithm.
- **Example** Any *hash function* with more inputs than outputs will necessarily have collision.

Summary

- Inclusion-exclusion principle $|A \cup B| = |A| + |B| - |A \cap B|$
- Counting by enumeration: list all possible outcome, draw a tree to help visualize
- Sum rule as special case of inclusion-exclusion principle $|\mathcal{P} \cup \mathcal{Q}| = |\mathcal{P}| + |\mathcal{Q}|$
- Product rule as application of Cartesian product $|\mathcal{P} \times \mathcal{Q}| = |\mathcal{P}| \cdot |\mathcal{Q}|$
- Subtraction rule as application of complement: $A \subset \mathcal{S}$ then for $A^c \setminus \mathcal{S}$, we have $|A^c| = |\mathcal{S}| - |A|$
- Floor function and counting number of divisible integers
- A permutation is an ordered arrangement of a set of objects
- The number of permutations of n distinct objects is $n! := n(n-1)(n-2)\cdots 1$ factorial
- Generalized permutation of n objects, possible non-distinct is $\dfrac{n!}{n_1! \ldots n_r!} = \dbinom{n}{n_!, n_2, ..., n_r}$ division rule
- A combination is an unordered arrangement of a set of objects
- Binomial coefficient n-choose-k without repetition is $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$
- n-choose-k with repetition is $\dbinom{n+k-1}{k}$
- Counting using binomial and multinomial coefficients
- Counting via bijection
- Counting via generating function
- Solving recursion by generating function
- Pigeonhole principle