#### COMP1311 Combinatorics

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1st draft May 24, 2023 Counting by set theory Inclusion-exclusion principle Enumeration, sum rule & product rule Counting by binomial coefficient Permutation, factorial and division rule Combination and binomial **Binomial** expansion Trinomial & multinomial Advanced counting techniques Counting via bijection Counting via generating function / z-transform Recursion, partial fraction and generating function Pigeonhole principle Advanced<sup>2</sup> counting techniques Gosper's algorithm Hypergeometric function

#### Information

- Combinatorics = "counting"
- Warning: counting is hard

the hardest area in mathematics it is universal, important for computer science

• Study: slide + madbook + reading + video

self learning

- Discrete Mathematics and Its Applications by Kenneth Rosen, enough for this
- Concrete mathematics: a foundation for computer science by Graham, Knuth & Patashnik, classic
- Combinatorics Through Guided Discovery by Bogart, free book
- Schaum's Outline of Combinatorics for more practice problems
- Counting: The art of enumerative combinatorics by Martin, nice
- Enumerative Combinatorics Volume 1 by Richard Stanley, advanced
- 102 Combinatorial Problems From the Training of USA IMO Team
- A stackexhcange post on advanced books if plan to do a PhD

#### Prerequisite: Set theory (I assume you know)

- $X := \{x \mid \text{set description}\}$  set
- universal set U and empty set  $\varnothing$  largest/smallest sets
- $x \in X, x \notin X$  membership
- $X^c := \{x \mid x \notin X\}$  complement
- $\bullet \ |X| \coloneqq \#\{x \, | \, x \in X\} \qquad \qquad \mathsf{cardinality}$ 
  - X is finite if  $|X| < +\infty$  finite set
- $X \cup Y \coloneqq \left\{ x \, | \, x \in X \text{ OR } x \in Y \right\}$  union
- $\bullet \ X \cap Y \coloneqq \big\{ x \, | \, x \in X \text{ AND } x \in Y \big\} \qquad \qquad \text{intersection}$
- $X \setminus Y := \left\{ x \, | \, x \in X \text{ AND } x \notin Y \right\}$  complement
- $\bullet \ X \times Y \coloneqq \big\{ (x,y) \, | \, x \in X \text{ AND } y \in Y \big\} \ \text{Cartesian prod}.$
- X, Y disjoint iff  $X \cap Y = \emptyset$ 
  - $\bullet \ |X\cup Y|=|X|+|Y|$
- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$  natural number, integer, reals
- Function: inject, surject, biject

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Advanced<sup>2</sup> counting techniques Gosper's algorithm Hypergeometric function

#### Inclusion-exclusion principle

 $|X \cup Y| = |X| + |Y| \iff X, Y$  disjoint

• Two sets:  $|A \cup B| = |A| + |B| - |A \cap B|$ 

#### Proof

 $(*) - (\dagger)$  gives

$$\begin{aligned} |A \cup B| &= |A \cup (B \setminus A)| & \stackrel{\text{disjoint}}{=} |A| + |B \setminus A| \quad (*) \\ |B| &= |(B \setminus A) \cup (B \cap A)| & \stackrel{\text{disjoint}}{=} |B \setminus A| + |B \cap A| \quad (\dagger) \end{aligned}$$

• Three sets:  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ . **Proof**: recursion (use the previous case)  $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)| = |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$ . By distributive property  $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ 

 $|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$ = |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|]

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|,$$

where  $|A \cap B \cap A \cap C| \stackrel{\text{commutative}}{=} |A \cap A \cap B \cap C| \stackrel{\text{idempotent}}{=} |A \cap B \cap C|.$ 

#### Inclusion-exclusion principle of four sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
(2)

• **Proof**: recursion (use the previous cases)

 $-|C \cap D|$ 

#### A pattern

0 set  $|\emptyset| = 0$ . 1 set |A| = |A|. 2 sets  $|A \cup B| = |A| + |B| - |A \cap B|$ . 3 sets  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$ 4 sets  $|A \cup B \cup C \cup D| =$  $+|A \cap B \cap C|$  $-|A \cap B \cap C \cap D|$ +|A| $-|A \cap B|$ +|B| $-|A \cap C|$  $+|A \cap B \cap D|$ +|C| $-|A \cap D|$  $+|A \cap C \cap D|$ +|D| $-B \cap C$  $+ B \cap C \cap D$  $-B \cap D$  $-|C \cap D|$ 

	1-set	2-set	3-set	4-set
	1			
$ A \cup B $	2	1		
$ A \cup B \cup C $	3	3	1	
$ A\cup B\cup C\cup D $	4	6	4	1

Multiplying polynomial			power-1	power-2	power-3	power-4
Consider $(1+x)^2 = 1x^0 + 2x^1 + 1x^2$ coefficients	$1, 2, 1\}$ (1 + a)	$(r)^{1}$	1			
Consider $(1+x)^3 = 1x^0 + 3x^1 + 3x^2 + 1x^3$ coefficients [1]	(1 + a)	$(v)^2$	2	1	1	
$\frac{1}{1} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{12} + \frac{1}{12} +$	(1 + a)	$\frac{v}{4}$	3	3	1	1
Consider $(1+x)^4 = 1x^0 + 4x^1 + 6x^2 + 4x^3 + 1x^4$ coefficients $\{1, 4, 4\}$	$6, 4, 1\}$ (1 + 3	()	4	0	4	1

- · Later we will see that these are binomial coefficients.
- · We can count things using the coefficients of polynomials (they have a bijection)

# What's the big deal of inclusion-exclusion principle? Not in exam

• Theorem (Generalized inclusion-exclusion principle) For any finite sequence  $A_1, \ldots, A_n$  of  $n \ge 2$  subsets of a finite set X, we have

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{\substack{I \subseteq \{1,\dots,n\}\\I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

• Application 1: The number of surjections,  $S_{np}$ , between a *n*-set A and a *p*-set B, where  $n \ge p$ , is

$$S_{np} = p^{n} - {\binom{p}{1}} (p-1)^{n} + {\binom{p}{2}} (p-2)^{n} + \dots + (-1)^{p-1} {\binom{p}{p-1}}.$$

- Application 2: chromatic polynomial in graph coloring problem
- Application 3: Stirling numbers of the second kind
- Application 4: counting derangement

# Counting by enumeration

- Enumeration = list all possible outcomes
- E.g. how many ways to form a length-1 string from alphabet  $\{1, 2, 3, 4, 5, 5, 6, 7\}$ ?

$$\mathcal{S} = \{1, 2, 3, 4, 5, 5, 6, 7\} \implies |S| = 7.$$

• E.g. how many ways to form a length-3 string from alphabet  $\{0, 1\}$ ?

$$S = \{000, 001, 010, 100, 011, 101, 110, 111\} \implies |S| = 8.$$

• E.g. how many ways to form a length-3 string from alphabet  $\{A, B, C\}$ ?

 $S = \{AAA, AAB, ABA, BAA, AAC, ACA, CAA, ABC, BAC, oh no \dots\}$ 

we need a better way to do this.

# **Tree & multiplication**



$$|S| = 7^4 = 2401.$$

Before using tree-thinking, you may not even know how to solve this question!

#### Product rule: application of Cartesian product



- $\mathcal{P} \times \mathcal{Q} =$  the Cartesian product of  $\mathcal{P}, \mathcal{Q}$
- $|\mathcal{P} \times \mathcal{Q}| = |\mathcal{P}| \cdot |\mathcal{Q}|$
- E.g. A daily diet consists of a breakfast selected from B, a lunch from L, a dinner from D:

$$B = \{pancakes, bacon and eggs, bagel\}$$
$$L = \{burger and fries, salad, macaroni, pizza\}$$
$$D = \{steak, pasta\}$$

Then the set of all possible daily diets  $= B \times L \times D$ 

- #possible diets =  $|B \times L \times D| \stackrel{\text{prod. rule}}{=} |B| \cdot |L| \cdot |D| = 3 \cdot 4 \cdot 2 = 24$
- By tree: a 3-branch  $\times$  4-branch  $\times$  2-branch tree with 24 edges
- By enumeration: list all the 24 choices

### Product rule, examples

- E.g. How many length-4 strings over the alphabet  $\{0, 1, ..., 9\}$  do not begin with 0?
- Let  $A_i$  be the set of possible alphabet at the *i*th digit, then

# strings = 
$$|A_1 \times A_2 \times A_3 \times A_4| \stackrel{\text{product rule}}{=} |A_1| \cdot |A_2| \cdot |A_3| \cdot |A_4| = 9000$$

- $|A_1| = 9$  (::  $A_1 = \{1, 2, \dots, 9\}$ )
- $|A_2| = |A_3| = |A_4| = 10$

There are 9000 length-4 strings do not begin with 0. This is the #4-digit positive integers with no leading 0.

- E.g. How many length-2 strings over the alphabet  $\{0, 1, ..., 9\}$  do not have repeated digits?
- $|A_1 \times A_2|$
- $|A_1| = 10$
- $|A_2| = 9$  choices (1 less than than  $|A_1|$  to avoid repetition)
- E.g. How many length-4 strings over the alphabet  $\{0, 1, ..., 9\}$  do not have repeated digits?
- $|A_1 \times A_2 \times A_3 \times A_4| = 10 \cdot 9 \cdot 8 \cdot 7$

#### Sum rule, a special case of incl-excl principle $|A \cup B| = |A| + |B| - |A \cap B|$

• E.g. In cafe, you have soup xor salad. Menu: 2 soups, 4 salads. How many choices do you have?

$$|\mathsf{soup} \cup \mathsf{salad}| \stackrel{\mathsf{incl-excl.}}{=} \underbrace{|\mathsf{soup}| + |\mathsf{salad}|}_{\mathsf{either soup or salad}} - \underbrace{|\mathsf{soup} \cap \mathsf{salad}|}_{=0 \ \because \ \mathsf{not} \ \mathsf{both}} = 2 + 4 = 6$$

- E.g.  $A = \{1\}, B = \{2, 3\}, C = \{3, 4\}$ , what is  $|A \cup B \cup C|$ ?
  - Approach 1: use the inclusion-exclusion principle for 3 sets

$$\begin{split} \left| A \cup B \cup C \right| & \stackrel{\mathsf{incl-excl.}}{=} & \left| A \right| + \left| B \right| + \left| C \right| - \left| A \cap B \right| - \left| A \cap C \right| - \left| B \cap C \right| + \left| A \cap B \cap C \right| \\ & = & 1 + 2 + 2 - \left| \varnothing \right| - \left| \varnothing \right| - \left| \{3\} \right| + \left| \varnothing \right| \\ & = & 1 + 2 + 2 - 0 - 0 - 1 + 0 = 4 \end{split}$$

• Approach 2

$$\begin{split} \left| A \cup B \cup C \right| \; = \; \left| A \cup (B \cup C) \right| & \stackrel{\text{incl-excl.}}{=} & \left| A \right| + \left| B \cup C \right| - \left| A \cap (B \cup C) \right| \\ & = & 1 + \left| B \right| + \left| C \right| - \left| B \cap C \right| + \left| \varnothing \right| \\ & = & 1 + 2 + 2 - \left| \left\{ 3 \right\} \right| + 0 \\ & = & 5 - 1 \; = \; 4 \end{split}$$

• Approach 3: construct  $A \cup B \cup C$ .

This is the most expensive approach if the sets has many elements (you have to check one by one).

#### Subtraction rule, complement and power set

• Subtraction rule: if  $A \subset S$ , then for complement  $A^c := S \setminus A$ , we have

$$|A^{c}| = |\mathcal{S}| - |A|.$$
 (Subtraction rule)

- E.g. How many subsets of  $X = \{1, 2, 3\}$  contain at least 2 (including 2) elements?
- Let  $S \subset X$  that contains at least 2 (including 2) elements.
- Let  $P = 2^X$  be the power set (set of all possible subsets) of X

$$P = \left\{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\right\}$$

$$\begin{split} S &= P \setminus S^c \\ &= P \setminus \{\text{set containing 1 or 0 element}\} \\ &= P \setminus \{\text{set containing 1 element } \cup \text{ set containing 0 element}\} \\ &\stackrel{\{\}=\varnothing}{=} P \setminus \left(\{1\} \cup \{2\} \cup \{3\} \cup \{\}\right) \\ &|S| \stackrel{\text{subtraction rule}}{=} |P| - \left(|\{1\} \cup \{2\} \cup \{3\}| + |\{\}|\right) \\ &= 2^3 - (3 + 1) = 4 \end{split}$$

Floor function:  $\mathbb{R} \to \mathbb{Z} \mid x \rfloor \coloneqq \max \left\{ m \in \mathbb{Z} \mid m \le x \right\}$ 

Example

- |2| = 2
- |2.4| = 2
- |2.0000000001| = 2
- |2.99999999999| = 2
- |-2| = -2
- |-2.00000001| = -3
- |-2.999999999| = -3
- $|\pi| = 3$
- $|-\pi| = -4$

Ceiling function  $\lceil x \rceil \coloneqq \min \left\{ m \in \mathbb{Z} \mid m \ge x \right\}.$ 



#### Example of Incl-excl principle in number theory: #divisible integers

• E.g. <sup>1</sup> How many integers n for  $1 \le n \le 2001$  are divisible by 3 or 4 but not 5?

Ans: 801.

- E.g.  $N := \{n \in \mathbb{N} \mid n \leq 20\}$ . How many integers in N are divisible by either 2 or 3?
- Let A,B be the set of integers in N that are divisible by 2 and by 3, respectively.
- $|A| = \lfloor \frac{20}{2} \rfloor = 10$ ,  $|B| = \lfloor \frac{20}{3} \rfloor = \lfloor 6.666 \rfloor = 6$
- $A \cap B$  is the set of integers in N that are by 2 AND 3 (=6), and  $|A \cap B| = \lfloor \frac{20}{6} \rfloor = \lfloor 3.333 \rfloor = 3$
- There are  $|A \cup B| = |A| + |B| |A \cap B| = 10 + 6 3 = 13$  integers in N that are by 2 or 3.
- E.g. How many integers n where  $1 \le n \le 100$  are **NOT** divisible by 2, 3, 5?
- Let A, B, C be the set of n divisible by 2, by 3, by 5, respectively.
- Number of integers n divisible by at least one 2, 3, 5 is

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 3} \right\rfloor - \left\lfloor \frac{100}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{100}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{100}{2 \cdot 3 \cdot 5} \right\rfloor \\ &= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74 \end{aligned}$$

• By complement, the number of integers n **NOT** divisible by 2, 3, 5 is

$$|\{1, 2, ..., 100\}| - |A \cup B \cup C| = 100 - 74 = 26.$$

<sup>&</sup>lt;sup>1</sup>American Mathematics Competitions 2001

#### Product rule & sum rule: counting passwords

- E.g. A valid password is a sequence of between 6 and 8 symbols. The first symbol must be a letter (lowercase or uppercase), the remaining symbols can be letters or digits. How many possible passwords?
- Define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

 $F = \{a, b, c, \dots, z, A, B, C, \dots, Z\}, \qquad S = \{a, b, c, \dots, z, A, B, C, \dots, Z, 0, 1, \dots, 9\}$ 

- The Cartesian product set  $S^2 := S \times S$  is the set of length-2 strings  $\{\alpha \beta \mid \alpha \in S, \beta \in S\}$
- Three possible cases of passwords:  $F \times S^5_{c}$  length-6 password

 $F \times S^6$  length-7 password  $F \times S^7$  length-8 password

The set of all possible passwords  $(F\times S^5)\,\cup\,(F\times S^6)\,\cup\,(F\times S^7),$  with cardinality

$$\begin{split} \left| (F \times S^5) \cup (F \times S^6) \cup (F \times S^7) \right| &= \left| F \times S^5 \right| + \left| F \times S^6 \right| + \left| F \times S^7 \right| & \text{sum rule (incl-excl principle)} \\ &= \left| F \right| \cdot \left| S \right|^5 + \left| F \right| \cdot \left| S \right|^6 + \left| F \right| \cdot \left| S \right|^7 & \text{product rule} \\ &= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 & \cong 1.8 \times 10^{14} \end{split}$$

 $1.8 \times 10^{14}$  passwords. The probability of "correct guess in one trial" is  $\frac{1}{1.8 \times 10^{14}} = 5 \times 10^{-15}$ .

You are  $10^8$  times more likely to be hit by thunder.

- If you have 6 chances to guess the password.
  - The probability of all 6 guesses are wrong:  $(1 5 \times 10^{-15})^6$
  - The probability of at least one guess is correct is  $1 (1 5 \times 10^{-15})^6 = 3 \times 10^{-14}$

You are  $10^7 \ {\rm times}$  more likely to be hit by thunder.

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Advanced<sup>2</sup> counting techniques Gosper's algorithm Hypergeometric function

## (Not in exam) We count things using integer, but actually what is integer? Definition of N by Zermelo-Fraenkel Set Theory. Notation by von Neumann.



Important:  $\varnothing = \{\}$ , and  $A = \varnothing, B = \{\varnothing\}$  and  $C = \{\{\varnothing\}\}$  are different things. |A| = 0, |B| = |C| = 1

# (Not in exam) There are two 'kinds" of integer

- If you need numbers to count "how many items do I have", then you are using cardinal numbers.
- If you need to make a list, or a enumeration, then you are using ordinal.
- Ordinal Numbers: Indicate position or order.
  - Examples: 1st, 2nd, 3rd, etc.
  - Used to describe the position of elements in a sequence.
  - Example: In the list {apple, banana, cherry}, "banana" is the 2nd item.
- Cardinal Numbers: Indicate quantity or count.
  - Examples: 1, 2, 3, etc.
  - Used to describe the number of elements in a set.
  - Example: There are 3 fruits in the list {apple, banana, cherry}.
- Ordinal and cardinal are NOT the same thing

## Permutation

Under permutation, ordinal changes but cardinal stays the same



- There are three boxes  $\iff$  the cardinal is 3  $\iff$  the cardinality is 3
- The 1,2,3 are label / position / index / address of the boxes
- The a,b,c are content of the box
- in fact we don't care about the content: you can replace a,b,c by names, people, books, etc
- therefore if we care about ordinal
- Under permutation  $\pi$ , the ordinal changes but cardinal stays the same
- $\pi(1) = 3$
- $\pi(2) = 3$
- $\pi(3) = 1$

#### Make sure you understand this slide

- Let set  $S=\{1,2,3\}$
- The cardinality of S, denoted by |S|, is 3
- One of the permutation of S is  $\pi(S) = \{3, 1, 2\}$
- All the possible permutations of S is a set of set

$$\operatorname{Sym}(3) = \begin{cases} \{1, 2, 3\}\\ \{1, 3, 2\}\\ \{2, 1, 3\}\\ \{2, 3, 1\}\\ \{3, 1, 2\}\\ \{3, 2, 1\} \end{cases}$$

we call this set the symmetric group of S, denoted as Sym(S) or Sym(3)

- Asking "how many permutation are there for S" is actually asking "what is the cardinality of Sym(3)
- No worry we are not going to talk about abstract algebra

 $\mathbf{Sym}(4)$ 



Fancy figures in <a href="https://en.wikiversity.org/wiki/Symmetric\_group\_S4">https://en.wikiversity.org/wiki/Symmetric\_group\_S4</a> No worry we are not going to talk about abstract algebra

# Permutation & factorial

• Def (Permutation)

A permutation (denoted by  $\pi$ ) is an ordered arrangement of a set of **distinct** objects.

• E.g. 
$$S = \{1, 2, 3, 4, 5\}$$
 and a permutation  $\pi(S) = \{2, 5, 4, 3, 1\}$   
 $\pi(1) = 5 = \text{put 1 to 5}$   
 $\pi(3) = 4 = \text{put 3 to 4}$ 

- Permutation is not unique:  $|Sym(n)| \neq 1$  unless n = 1
- Theorem The number of permutations of a *n*-set of distinct objects is  $n(n-1)\cdots 2\cdot 1$ . **Proof**. Tree / induction.
- E.g. How many permutations can be formed from the letters HAPY?  $4 \cdot 3 \cdot 2 \cdot 1 = 24$
- Recursive definition of factorial

$$k! = \begin{cases} 1 & k = 0 \text{ base case} \\ k(k-1)! & k > 0 \end{cases}$$
(factorial)

Why 0! = 1? Google it yourself.

Not in exam

- Gamma function generalizes factorial. E.g.,  $\frac{1}{2}! = \frac{\sqrt{\pi}}{2}$
- We focus on finite combinatorics (numbers and sets are finite)
- Crazy things happen at infinity, e.g.,  $\infty! = 1 \cdot 2 \cdot 3 \cdot ... = \sqrt{2\pi}$ , this is analytic continuation, like  $i = \sqrt{-1}$ . 23/118

# Generalized permutation & division rule: permutation with non-distinct objects

• Def (Permutation)

A permutation is an ordered arrangement of a set of distinct objects.

• Def (Generalized permutation)

A Generalized permutation is an ordered arrangement of a set of non-distinct objects.

- E.g. How many permutations can be formed from the letters HAPPY?
  - First, put subscript  $HAP_1P_2Y$  and treat it as 5 distinct letters.
  - 5 distinct letters so 5! = 120 permutations.
  - But P<sub>1</sub> and P<sub>2</sub> are the same letter, we double-counted some of these 120 permutations. E.g.

 $\begin{array}{rl} \mathsf{HAP_1P_2Y}, & \mathsf{HAP_2P_1Y} & \text{are both HAPPY} \\ \mathsf{P_1HAYP_2}, & \mathsf{P_2HAYP_1} & \text{are both PHAYP} \\ \mathsf{So \ correct \ number \ of \ generalized \ permutations \ is \ } \frac{120}{2!} = 60 \end{array}$ 

• Theorem The number of permutations of a set of n elements, possibly non-distinct, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$
 (Division rule)

where  $n_1, n_2, \ldots, n_r$  are number of alike objects.

• If all objects are distinct,  $n_1 = n_2 = \cdots = n_r = 1$  and  $\frac{n!}{n_1!n_2!\cdots n_r!}$  reduces to n!.

 $24 \, / \, 118$ 

# Example (1/3) Find the number of generalized permutations of (1, 1, 2, 3, 4)

• First assume the two 1's are distinct, so we have 5! ways to permute  $(1_a, 1_b, 2, 3, 4)$ But in these 5! ways, we double-counted some cases:

$$(1, 1, 2, 3, 4) \begin{cases} (1_a, 1_b, 2, 3, 4) \\ (1_b, 1_a, 2, 3, 4) \end{cases} \\ (1, 2, 1, 3, 4) \begin{cases} (1_a, 2, 1_b, 3, 4) \\ (1_b, 2, 1_a, 3, 4) \end{cases}$$

Every 2! of our 5! ways corresponds to one actual generalized permutation. So the # of generalized permutations of (1,1,2,3,4) is

$$\frac{5!}{2!}$$

- For  $\frac{n!}{n_1!n_2!\cdots n_r!}$ 
  - n = 5: we have 5 things to permute
  - r = 1: 1 type of alike objects
  - $n_1 = 2$ : for type-1 objects, there are 2 of them

# Example (2/3) Find the number of generalized permutations of (1, 1, 1, 2, 3).

• First assume the three 1's are distinct, so we have 5! ways to permute  $(1_a, 1_b, 1_c, 2, 3)$ But in these 5! ways, we over-counted some cases:

$$(1, 1, 1, 2, 3) \begin{cases} (1_a, 1_b, 1_c, 2, 3)\\ (1_a, 1_c, 1_b, 2, 3)\\ (1_b, 1_a, 1_c, 2, 3)\\ (1_b, 1_c, 1_a, 2, 3)\\ (1_c, 1_a, 1_b, 2, 3)\\ (1_c, 1_b, 1_a, 2, 3) \end{cases}$$

Every 3! = 6 of our initial 5! ways corresponds to one actual permutation. So the # of permutations of (1, 1, 1, 2, 3) is

$$\frac{5!}{3!}$$

• For  $\frac{n!}{n_1!n_2!\cdots n_r!}$ 

- n = 5: we have 5 things to permute
- r = 1: 1 type of alike objects
- $n_1 = 3$ : for type-1 objects, there are 3 of them

### Example (3/3) Find the number of generalized permutations of (1, 1, 1, 2, 2)

First assume the three 1's and the two 2's are distinct, so we have 5! ways to permute  $(1_a, 1_b, 1_c, 2_a, 2_b)$ But in these 5! ways, we over-counted some cases:

$1, 1, 2, 2) \$	$ \begin{pmatrix} (1_a, 1_b, 1_c, 2_a, 2_b) \\ (1_a, 1_c, 1_b, 2_a, 2_b) \\ (1_b, 1_a, 1_c, 2_a, 2_b) \\ (1_b, 1_c, 1_a, 2_a, 2_b) \\ (1_c, 1_a, 1_b, 2_a, 2_b) \\ (1_c, 1_b, 1_a, 2_a, 2_b) \\ (1_a, 1_b, 1_c, 2_b, 2_a) \\ (1_a, 1_c, 1_b, 2_b, 2_a) \\ (1_b, 1_a, 1_c, 2_b, 2_a) \\ (1_b, 1_c, 1_a, 2_b, 2_a) \\ (1_c, 1_a, 1_b, 2_b, 2_b, 2_b) \\ (1_c, 1_b, 2_b, 2_b, 2_b, 2_b) \\ (1_c, 1_b, 2_b, 2_b, 2_b, 2_b, 2_b, 2_b, 2_b, 2$
	$\binom{(1_c, 1_a, 1_b, 2_b, 2_a)}{(1_c, 1_b, 1_a, 2_b, 2_a)}$

Every 3!2! = 12 of our initial 5! ways corresponds to one actual permutation. Num of generalized permutations of (1, 1, 1, 2, 3) is  $\frac{5!}{3!2!}$ 

(1

# Now you understand what is $\frac{n!}{n_1!n_2!\cdots n_r!}$

 $\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{(\mathsf{num of objects})!}{(\mathsf{num of type-1 objects})!(\mathsf{num of type-2 objects})!\cdots(\mathsf{num of type-r objects})!}$ 

- $n_1 + n_2 + \cdots + n_r \leq n$ It is impossible for the sum  $n_1 + n_2 + \cdots + n_r$  to exceed n.
- In fact, such expression is the multinomial coefficient

$$\frac{n!}{n_1!n_2!\cdots n_r!} =: \binom{n}{n_1, n_2, \dots, n_r}$$

- E.g. How many permutation for the string "baby"?
- type-1 letter b has 2 occurrence: divided by 2!
- type-2 letter a has 1 occurrence: divided by 1!
- type-3 letter y has 1 occurrence: divided by 1!
- r=3 and n=4 here
- Solution  $\frac{4!}{2!1!1!} = 12$

 $\frac{n!}{n_1!n_2!\cdots n_r!}$  = # generalized permutations of a *n*-set with repeated elements

• E.g. A chess tournaments has 10 competitors, of which

4 are from Russia 3 are from US 2 are from UK 1 is from Brazil

If the tournament result lists the nationalities of the players in the order in which they placed, how many outcomes are possible?

- 10 competitors, so 10! permutations
- 4 Russians, so 4! repeated permutations to be divided
- 3 US, so 3! repeated permutations to be divided
- 2 UK, so 2! repeated permutations to be divided
- 1 Brazil, so 1! repeated permutations to be divided

 $\frac{10!}{4!3!2!1!} = 12600$ 

# (Not in exam) Derangement

• Def (Derangement)

A permutation that no element appears in its original position.

- A subset of symmetric group
- For  $\{1, 2, 3, 4\}$ , the number of derangement is 9.
- The number of derangement, denoted by !n, has the formula

$$!n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right) = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

How do you prove this: Incl-excl principle.

• Example.

$$!4 = 4! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 9.$$

1234	2314	3412
1243	2341	3421
1324	2413	4123
1342	2431	4132
1423	3124	4213
1432	3142	4231
2134	3214	3412
2143	3241	4321

# Combination

- Combination vs Permutation = commutative vs not commutative
  - An relation R is commutative if aRb = bRa.
  - A permutation is an ordered arrangement
  - A combination is an **unordered** arrangement
  - Boxing analogy: you fight Elon Musk, Joe Biden, Mike Tyson
    - Permutation: the order (Mike Tyson, Elon Musk, Joe Biden)  $\neq$  (Joe Biden, Elon Musk, Mike Tyson)
    - Combination: the order (Mike Tyson, Elon Musk, Joe Biden) = (Joe Biden, Elon Musk, Mike Tyson)
- E.g. A committee of 3 is to be formed from a group of 5 people (1, 2, 3, 4, 5). How many different committees are possible?



# **Binomial coefficient & number of combinations**

• Def (*n*-choose-*k*)

Given integers  $n \ge k \ge 0$ , the *n*-choose-*k*, denoted as  $\binom{n}{k}$ , is defined as  $\frac{n!}{(n-k)!k!}$ .

• E.g. 5-choose-3 is  

$$\begin{pmatrix} 5\\3 \end{pmatrix} = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$$

$$\begin{pmatrix} 4\\2 \end{pmatrix} = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

- Meaning of binomial coefficients
  - $\bullet\,$  number of combinations in  $n\mbox{-}{\rm choose-}k$  / number of  $k\mbox{-}{\rm sets}$  in  $n\mbox{-}{\rm set}$
  - number of bit strings from length n with exactly k 1's
  - the coefficient of  $x^{\overline{k}}y^{n-k}$  in  $(x+y)^n$

• If 
$$k < 0$$
 or  $k > n$ ,  $\binom{n}{k}$  is meaningless and defined as 0.

		power-1 term	power-2 term	power-3 term	power-4 term	
	$(1+x)^1$	1				
	$(1+x)^2$	2	1			
	$(1+x)^3$	3	3	1		
	$(1+x)^4$	4	6	4	1	
is the sar	is the same as					
		power-1 term	power-2 term	power-3 term	power-4 term	
	$(1+x)^1$	$\begin{pmatrix} 1\\1 \end{pmatrix}$				
	$(1+x)^2$	$\begin{pmatrix} 2\\1 \end{pmatrix}$	$\begin{pmatrix} 2\\ 2 \end{pmatrix}$			
	$(1+x)^3$	$\begin{pmatrix} 3\\1 \end{pmatrix}$	$\begin{pmatrix} 3\\2 \end{pmatrix}$	$\begin{pmatrix} 3\\ 3 \end{pmatrix}$		
	$(1+x)^4$	$\begin{pmatrix} 4\\1 \end{pmatrix}$	$\begin{pmatrix} 4\\2 \end{pmatrix}$	$\begin{pmatrix} 4\\ 3 \end{pmatrix}$	$\begin{pmatrix} 4\\4 \end{pmatrix}$	

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# Properties of binomial coefficient from Pascal's $\Delta$

•  $\sum_{k=1}^{n} \binom{n}{k} = 2^{n}$ 

•  $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$ 



sum of elements in the nth row

also the number of subsets of a n-element set

• 
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$
 equivalently  $\underbrace{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}_{a \text{ recurrence relation}}$  sum of consecutive (Can you see that  $\uparrow$  is a sum rule?)

sum of half of Pascal's  $\Delta$  is the next Fibonacci number

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \qquad n \ge k \ge 0. \qquad factorial expansion$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n \\ n-k \end{pmatrix}, \qquad integer n \ge 0, \qquad symmetry$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = \frac{r}{k} \begin{pmatrix} r-1 \\ k-1 \end{pmatrix}, \qquad integer k. \qquad absorption/extraction$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} r-1 \\ k-1 \end{pmatrix} + \begin{pmatrix} r-1 \\ k-1 \end{pmatrix}, \qquad integer k. \qquad addition/induction$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = (-1)^k \begin{pmatrix} k-r-1 \\ k-1 \end{pmatrix}, \qquad integer k. \qquad addition/induction$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = (-1)^k \begin{pmatrix} k-r-1 \\ k \end{pmatrix}, \qquad integer k. \qquad addition/induction$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} r-k \\ m-k \end{pmatrix}, \qquad integer k. \qquad addition/induction$$

$$\begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} r-k \\ m-k \end{pmatrix}, \qquad integer sm, k. \qquad trinomial revision$$

$$\sum_{k \in n} \begin{pmatrix} r+k \\ k \end{pmatrix} = \begin{pmatrix} r+n+1 \\ n \end{pmatrix}, \qquad integer n. \qquad parallel summation$$

$$\sum_{0 \le k \le n} \begin{pmatrix} k \\ m \end{pmatrix} = \begin{pmatrix} n+1 \\ m+1 \end{pmatrix}, \qquad integer n. \qquad vandermonde convolution$$

$$\sum_{k \in n} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} s \\ n-k \end{pmatrix} = \begin{pmatrix} r+s \\ n \end{pmatrix}, \qquad integer n. \qquad Vandermonde convolution$$

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Proving  $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ Algebraic proof

$$\binom{n}{k} + \binom{n}{k+1}$$

$$= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!}$$

$$= \frac{n!}{(n-k)!k!} \frac{k+1}{k+1} + \frac{n!}{(n-k-1)!(k+1)!} \frac{n}{n}$$

$$= \frac{n!(k+1)}{(n-k)!(k+1)!} + \frac{n!(n-k)}{(n-k)!(k+1)!}$$

$$= \frac{n!(k+1+n-k)}{(n-k)!(k+1)!}$$

$$= \frac{(n+1)!}{((n+1)-(k+1))!(k+1)!} = \binom{n+1}{k+1}$$

Algebraic proof can be tedious. So we should consider combinatorial proof. **Combinatorial proof** (Not in exam)

- Find an appropriate object to be counted
- Count that object in some way
- Count that object in a different way

The two counting methods give the same result. The proof Let  $S = \{a, b, c, \cdots, z\}$  has n + 1 people.

• (1st counting) There are  $\binom{n+1}{k+1}$  ways to pick k+1 people in S

• (2nd counting) Suppose e is a VIP in S. Now, we we are doing the same thing: "selecting the k + 1 people in S". With respect to e, there are two possibilities.

- VIP is selected. Then we know we need to pick k more people (not k + 1 because e has been selected here) in the remaining n people (not n + 1 because e has been selected). There are <sup>n</sup><sub>k</sub> ways.
- $\bigvee$  iP is not selected. Then we need to pick k + 1 (because e is not selected so we still need to pick k + 1 people) people in the remaining n (not n + 1 because e has been rejected) people. There are  $\binom{n}{k+1}$  ways.

These two yes-no events are disjoint so  $\binom{n}{k} + \binom{n}{k+1}$ .

• Both counting methods are counting the same people, so

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$
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# (Not in exam) E.g. Combinatorial proof of $\binom{4n}{2} = \binom{2n}{2} + 2\binom{n}{2} + 5n^2$

- (Left-hand side)  $\binom{4n}{2}$  means in a bag of 4n balls, choose 2 balls.
- (Right-hand side) In the 4n balls, 2n are red, n are blue and n are green. To pick 2 balls, we have
- $\binom{2n}{2}$  all red balls2n-choose-2•  $\binom{n}{2}$  all blue ballsn-choose-2•  $\binom{n}{2}$  all green ballsn-choose-2•  $\binom{2n}{1}\binom{n}{1}$  one red one blue 2-ballproduct rule•  $\binom{2n}{1}\binom{n}{1}$  one red one green 2-ballproduct rule•  $\binom{n}{1}\binom{n}{1}$  one blue one green 2-ballproduct rule
- Now by sum rule

$$\binom{2n}{2} + \binom{n}{2} + \binom{n}{2} + \binom{2n}{1}\binom{n}{1} + \binom{2n}{1}\binom{n}{1} + \binom{n}{1}\binom{n}{1} = \binom{2n}{2} + 2\binom{n}{2} + 2n^2 + 2n^2 + n^2$$
$$= \binom{2n}{2} + 2\binom{n}{2} + 5n^2$$

- We are counting the same thing, so  $\binom{4n}{2} = \binom{2n}{2} + 2\binom{n}{2} + 5n^2$ 

# Binomial coefficient for counting combinations: no repetition

- $\binom{n}{k}$  ways to choose k elements from a n-set if repetitions of elements are NOT allowed
- **E.g.**  $S = \{1, 2, 3, 4, 5, 6\}.$
- How many subsets are there?
- How many 4-subsets are there?
- E.g. How many length-10 strings over alphabet  $\{0,1\}$  contain 6 or more 1's?
  - Let  $S_6$  be the set of strings containing six 1's
- We need to compute  $|S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_9 \cup S_{10}|$ , by inclusion-exclusion principle

$$\begin{vmatrix} S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_{10} \end{vmatrix} \stackrel{\text{sum rule}}{=} |S_6| + |S_7| + |S_8| + |S_9| + |S_{10}| \\ = \begin{pmatrix} 10 \\ 6 \end{pmatrix} + \begin{pmatrix} 10 \\ 7 \end{pmatrix} + \begin{pmatrix} 10 \\ 8 \end{pmatrix} + \begin{pmatrix} 10 \\ 9 \end{pmatrix} + \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

= 386

# $\binom{n+k-1}{k}$ Binomial coefficient for counting combinations: with repetition ... (1/2)

 $\binom{n}{k} = \binom{n+k-1}{k}$  ways to choose k elements from a n-set if repetitions of elements are allowed

- E.g. A farm has cats and dogs. How many ways can you select three pets to take home?
  - $F = \{1, 2\}, 1 = \mathsf{cat}, 2 = \mathsf{dog}.$
  - Enumerate all the  $2^3 = 8$  possible cases

111	112	121	122	$n^3 - 8$ cases
211	212	221	222	2 = 6 cases

- There are repetitions within the combination: e.g.  $\underline{11}2$  and  $\underline{222}$
- Combination is unordered, so 112 = 211. After removing the same combinations we have

111 112 122 122 4 cases 
$$4 = \binom{2+3-1}{3} = \binom{4}{3}.$$

- E.g. A farm has birds, cats and dogs. How many ways can you select three pets to take home?
- $F = \{1, 2, 3\}, 1 = \text{bird}, 2 = \text{cat}, 3 = \text{dog}$ . Enumerate all the  $3^3 = 27$  possible cases

	133	132	131	123	122	121	113	112	111
$27  \mathrm{cases}$	233	232	231	223	222	221	213	212	211
	333	332	331	323	322	321	313	312	311

- There are repetitions: e.g. <u>11</u>2 and <u>22</u>3
- Combination is unordered, so 133 = 313 = 331. After removing the same combinations we have

# $\binom{n+k-1}{k}$ Binomial coefficient for counting combinations: with repetition ... (2/2)

- E.g. A farm has birds, cats, dogs, ducks. How many ways can you select three pets?
  - $F = \{1, 2, 3, 4\}$ , 1=bird, 2=cat, 3 =dog, 4 =duck. Enumerate all the  $4^3 = 64$  possible cases

64 cases . . . 

- There are **repetitions**: e.g.  $\underline{11}2$  and  $\underline{22}3$
- Combination is unordered, so 133 = 313 = 331. After removing the same combinations we have

$$\binom{4+3-1}{3} = \binom{6}{3} = 20$$

• E.g. There are five types of batteries: AAA, AA, C, D, and 9-volt. How many ways can we choose the twenty batteries?

$$\binom{5+20-1}{20} = \binom{24}{20} = 10626.$$

• E.g. There are 5 types of soda. You buy 7 cans of soda. How many selections can you make?

$$\binom{5+7-1}{7} = \binom{11}{7} = 330$$

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# Combination and permutation together, example .. (1/2)

• E.g. In boxing, you fight against 3 boxers consecutively from a group (1, 2, 3, 4, 5). How many different fight orders are possible? 12345



 $\binom{5}{3}3!$ 

- In each group, there are 3! permutation, so by product rule, the total number of fight orders

• Generalization The number of permutation of k subset of a n-set is 
$$\binom{n}{k}k!$$
.

Some books denote this as  $P(n,k) = \frac{n!}{(n-k)!}$ 

# Combination and permutation together, example .. (2/2)

- E.g. From the 26-alphabet, how many length-5 strings, no repeated letters are allowed, are possible?  $\binom{26}{5}5! = 26^{\frac{5}{5}}$
- E.g. From the 26-alphabet, how many length-5 strings, with repeated letters are allowed among the chosen letters, are possible?
   <sup>26</sup>
   <sub>5</sub>
   <sub>5</sub>
- $\bullet\,$  E.g. From the 26-alphabet, how many length-5 strings, with repeated letters are allowed, are possible?  $26^5$
- E.g. From the 26-alphabet, how many length-5 strings, with the first two letters cannot be the same, are possible?  $\binom{26}{5}5 \cdot 4 \cdot 5 \cdot 5 \cdot 5 = \binom{26}{5} \cdot 4 \cdot 5^4$
- E.g. From the 26-alphabet, how many length-7 English names are possible ?  $26^7\,$
- E.g. From the 26-alphabet, how many length-7 English names start with letter a are possible ?  $26^7-25\cdot26^6$

Binomial theorem  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ 

**E.g.** 
$$(x+y)^3$$
  

$$= \binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0$$

$$= y^3 + 3xy^2 + 3x^2y + x^3.$$

 $\mathop{!\!!\!!}{[x^a]}f(x)$  means the coefficient of  $x^a$  in f(x)

• E.g. 
$$[x^5](1+2x)^9 = \begin{pmatrix} 9\\5 \end{pmatrix} \cdot 2^5$$
.  
• E.g.  $[x^5y^6](5x-3y)^{11} = \begin{pmatrix} 11\\5 \end{pmatrix} \cdot 5^5 \cdot (-3)^6$ .

• E.g. 
$$5^n = (3+2)^n = \sum_{k=0}^n \binom{n}{k} 3^k 2^{n-k}$$

To find  $[x^p y^q]$ , find the k  $(1 \le k \le n \in \mathbb{N})$  in the binomial expansion that give  $x^p y^q$ . If such k doesn't exist, there is no  $x^p y^q$  in the expansion.

• **E.g.** 
$$[x^3y^3]$$
 in  $(x^2 + y^2)^3$  is 0.

**Proof**. Let  $c = [x^3y^3](x^2 + y^2)^3$ . By binomial expansion

$$cx^{3}y^{3} = \underbrace{\binom{3}{k}(x^{2})^{k}(y^{2})^{3-k}}_{k \text{ th term in the expansion}} = \binom{3}{k}x^{2k}y^{6-2k}.$$

kth term in the expansion

Now comparing the indices:

Find the integer k that solves 3=2k AND 3-k=6-2k

There is no solution, so  $x^3y^3$  does not exist in  $(x^2+y^2)^3$  and  $c=0. \label{eq:constant}$ 

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# Mathematical Induction

- Simple induction
  - **Base case** Prove  $S_1$  is true.
  - Hypothesis step Assume  $S_m$  is true for some m
  - Induction step Using  $S_m$  to prove  $S_{m+1}$  is true.
- Strong induction
  - **Base case** Prove  $S_1$  and  $S_2$  are true.
  - Hypothesis step Assume  $S_m, S_{m+1}$  are true for some m
  - Induction step Using  $S_m$  and  $S_{m+1}$  to prove  $S_{m+2}$  is true.
- Generalized strong induction
  - **Base case** Prove  $S_1, S_2, \ldots, S_k$  are true.
  - Hypothesis step Assume  $S_m, S_{m+1}, ..., S_{m+k-1}$  are true for some m
  - Induction step Using  $S_m, S_{m+1}, ..., S_{m+k-1}$  to prove  $S_{m+k}$  is true.
- Related concept: well-ordering principle

# Mathematical Induction

- Fact: you can also prove them using
- partial fraction & telescoping
- Gosper's algorithm

#### Proving the binomial theorem via induction

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

- We will need identities  $\binom{p}{0} = \binom{q}{0}$  and  $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ .
- Proof. Base case If n = 1, we have  $(x + y)^1 = x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$ . Hypothesis step Assume the statement at n = m

$$(x+y)^m = \sum_{k=0}^m \binom{m}{k} x^k y^{m-k} \tag{H}$$

**Induction step** Consider the statement at m + 1, we have

$$\begin{split} & (x+y)(x+y)^m \stackrel{(\mathcal{H})}{=} (x+y) \sum_{k=0}^m \binom{m}{k} x^k y^{m-k} \\ & = (x+y) \Big( \binom{m}{0} x^0 y^m + \binom{m}{1} x^1 y^{m-1} + \dots + \binom{m}{m} x^m y^0 \Big) \\ & = \binom{m}{0} x^1 y^m + \binom{m}{1} x^2 y^{m-1} + \dots + \binom{m}{m-1} x^m y^1 + \binom{m}{m} x^{m+1} y^0 + \binom{m}{0} x^0 y^{m+1} + \binom{m}{1} x^1 y^m + \dots + \binom{m}{m} x^m y^1 \\ & = \binom{m}{0} x^0 y^{m+1} + \binom{m}{1} x^1 y^m + \binom{m+1}{1} x^1 y^m + \binom{m+1}{2} x^2 y^{m-1} + \dots + \binom{m+1}{m+1} x^{m+1} y^0 \\ & = \binom{m+1}{k} \binom{m+1}{k} x^k y^{m+1-k} \quad \Box \end{split}$$



Trinomial expansion  $(x+y+z)^3 = x^3+y^3+z^3+$ cross-terms

 $\frac{x^{3} + y^{3} + z^{3} + 3x^{2}y + 3x^{2}z + 3xy^{2} + 6xyz + 3xz^{2} + 3y^{2}z + 3yz^{2}}{2}$ 



# **Trinomial expansion**

 $\left| (p+q)^n \right| = \left| \sum_{k=0}^n \binom{n}{k} p^k q^{n-k} \right|$ 

• Cast trinomial as binomial

$$(x+y+z)^n = \left(x+(y+z)\right)^n \stackrel{\text{bi.thm}}{=} \sum_{k=0}^n \binom{n}{k} x^k (y+z)^{n-k} \stackrel{\text{bi.thm}}{=} \sum_{k=0}^n \binom{n}{k} x^k \sum_{j=0}^{n-k} \binom{n-k}{j} y^j z^{n-k-j}.$$

• Combine the binomial coefficients

$$\binom{n}{k}\binom{n-k}{j} = \frac{n!}{(n-k)!k!} \frac{(n-k)!}{(n-k-j)!j!} = \frac{n!}{k!j!(n-k-j)!} \xrightarrow{\text{rename } n-k-j=i} \frac{n!}{k!j!i!}$$

• Trinomial expansion

$$(x+y+z)^{n} = \sum_{\substack{i,j,k\\i+j+k=n}} \frac{n!}{i!j!k!} x^{k} y^{j} z^{i} = \sum_{\substack{i,j,k\\i+j+k=n}} \frac{n!}{i!j!k!} x^{i} y^{j} z^{k}.$$

 $\binom{n}{i,j,k} \coloneqq \frac{n!}{i!j!k!} \text{ with } i+j+k=n \text{ is the trinomial coefficient.}$ 

Trinomial expansion example  $(x + y + z)^4$ 

- $\frac{4!}{4!0!0!}x^4$
- $\frac{4!}{3!1!0!}x^3y^1$ ,  $\frac{4!}{3!0!1!}x^3z^1$
- $\frac{4!}{2!2!0!}x^2y^2$ ,  $\frac{4!}{2!1!1!}x^2y^1z^1$ ,  $\frac{4!}{2!0!2!}x^2z^2$
- $\frac{4!}{1!3!0!}x^1y^3$ ,  $\frac{4!}{1!2!1!}x^1y^2z^1$ ,  $\frac{4!}{1!1!2!}x^1y^1z^2$ ,  $\frac{4!}{1!0!3!}x^1y^0z^3$
- $\frac{4!}{0!4!0!}y^4$ ,  $\frac{4!}{0!3!1!}y^3z^1$ ,  $\frac{4!}{0!2!2!}y^2z^2$ ,  $\frac{4!}{0!1!3!}z^1z^3$ ,  $\frac{4!}{0!0!4!}z^4$ ,

 $x^{4} + 4x^{3}y + 4x^{3}z + 6x^{2}y^{2} + 12x^{2}yz + 6x^{2}z^{2} + 4xy^{3} + 12xy^{2}z + 12xyz^{2} + 4xz^{3} + y^{4} + 4y^{3}z + \cdots$ 

- How many terms: 3-choose-4 with repetition
- The coefficients are inside a Newton polytope
- The coefficients follows a lexicographic order

# Lexicographic order

• A total order such as  $\preceq$  is a binary relation that is

$$\begin{cases} \underbrace{a \leq a}_{\text{reflexive}} \\ \underbrace{((a \leq b) \land (b \leq c)) \implies a \leq c}_{\text{transitive}} \\ \underbrace{((a \leq b) \land (b \leq a)) \implies a = b}_{\text{antisymmetric}} \\ \underbrace{(a \leq b) \lor (b \leq a)}_{\text{total}} \end{cases}$$

- A lexicographic order is a way to compare strings based on the total order.
- A string is a sequence of alphabets.
- E.g. Alphabet =  $\{a, b, c\}$ , we consider string of length-3 with the total order:  $a \leq b \leq c$

$$aaa \preceq aab \preceq aac \preceq aba \preceq abb \preceq abc \preceq aca \preceq acb \preceq acc$$
$$\preceq baa \preceq bab \preceq bac \preceq bba \preceq bbb \preceq bbc \preceq bca \preceq bcb \preceq bcc$$
$$\preceq caa \preceq cab \preceq cac \preceq cba \preceq cbb \preceq cbc \preceq cca \preceq ccb \preceq ccc$$

You can think of this as "3-digit base-3" number system

#### Multinomial expansion

 $\left(x_1 + \dots + x_m\right)^n = \sum_{k_1, \dots, k_m \ge 0} \binom{n}{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m}.$  $k_1 + \dots + k_m = n$ **E.g.**  $(x + y + z + w)^4$ •  $\frac{4!}{4!0!0!0!}x^4$ •  $\frac{4!}{2!1!0!0!}x^3y^1$ ,  $\frac{4!}{2!0!1!0!}x^3z^1$ ,  $\frac{4!}{2!0!0!1!}x^3w^1$ •  $\frac{4!}{2!2!0!0!}x^2y^2$ ,  $\frac{4!}{2!1!1!0!}x^2y^1z^1$ ,  $\frac{4!}{2!1!0!1!}x^2y^1w^1$ ,  $\frac{4!}{2!0!2!0!}x^2z^2$ ,  $\frac{4!}{2!0!1!1!}x^2z^1w^1$ ,  $\frac{4!}{2!0!0!0!}x^2w^2$ , •  $\frac{4!}{1131010!}x^1y^3, \frac{4!}{1121110!}x^1y^2z^1, \frac{4!}{1121011!}x^1y^2w^1$  and so on ...  $4000 \succ 3100 \succ 3010 \succ 3001 \succ 2200 \succ 2110 \succ 2101 \succ 2020 \succ \ldots \succ 0004$ 

# Practice

• 
$$\left(x+2y+3z+4w\right)^5$$

• 
$$\left(-x+2y-3z+4w-5u\right)^3$$



# Special numbers (Not in exam but useful)

• Pochhammer falling factorial: 
$$k^{\underline{n}} = \frac{k!}{(k-n)!} = k(k-1)...(k-n+1)$$

• Pochhammer rising factorial: 
$$k^{\overline{n}} = \frac{(k+n-1)!}{(k-1)!} = k(k+1)...(k+n-1)$$

• Stirling numbers of the 1st kind:  $\begin{bmatrix} n \\ k \end{bmatrix} = \#$  ways to seat n people around k identical non-empty circular tables

$$x^{\underline{n}} = x(x-1)(x-2)...(x-n+1) = \sum_{k=0}^{n} {n \brack k} x^{k}$$
$$x^{\underline{3}} = x^{\underline{3}} - 3x^{2} + 2x \iff {3 \brack \underline{3}} = 1, {3 \brack \underline{2}} = -3, {3 \brack \underline{1}} = 2$$

• Stirling numbers of the 2nd kind:  ${n \\ k} \#$  of ways of putting n distinct balls into k identical boxes with at least one ball per box

#### Examples on $n^{\underline{r}}$

• Number of ordered selections with no repetition of r elements from a n-set

• 
$$n^{\underline{r}} = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

• E.g. 7 athletes in an Olympic event. How many ways can the gold, silver and bronze medals be awarded  $7^{\underline{3}}=7\cdot 6\cdot 5$ 

or

7-choose-3 winners  $\times$  permute-3 :  $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$  3!

• E.g. There are 6 kids in a party and I have 6 presents. How many ways can I give one preset to each kid?  $6^{\underline{6}} = 6 \cdot 5 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ 



• Number of unordered selections with no repetition of r elements from a n-set

• E.g. How many ways are there to pick a team of 4 from a group of 9?

• You buy a pizza. You can choose exactly 4 toppings from a list of 11 options. How many ways can you do this? Double topping is not ok here.  $\begin{pmatrix} 11 \\ 4 \end{pmatrix}$ 

 $\binom{n}{r}$ 

9)

# Examples on $\left< {n \atop r} \right>$

- Number of multisets of r elements, each element from a n-set
- Multiset: set with repeated element allowed
- E.g. A shop has three drinks: Coke, Pepsi and Water. How many way to buy 4 drinks?

$$\left\langle \begin{array}{c} 3\\4 \end{array} \right\rangle = \left( \begin{array}{c} 3+4-1\\4 \end{array} \right) = \left( \begin{array}{c} 6\\4 \end{array} \right)$$

Star-and-bar: you begin with

The job: place 4 stars inside two bars (two = 3-1)

 $\binom{n}{r}$ 

#### The four numbers, again

• Number of ordered selections with no repetition of r elements from a n-set

• Number of unordered selections with no repetition of r elements from a n-set

• Number of ordered sequence with repetition of r elements from a n-set

• Number of multisets of r elements, each element from a n-set

 $n^r$ 

n'

 $\binom{n}{r}$ 

#### Pizza madness: How many ways can you do this?

- You choose exactly 4 toppings from a list of 11 options. Double topping is not ok.
- You choose at most 4 toppings from a list of 11 options.
  - Double topping is not ok.
- You choose exactly 4 toppings from a list of 11 options.
- Double topping ok but only double, no triple.
- You choose at most 4 toppings from a list of 11 options.
- Double topping ok but only double, no triple.





#### Pizza madness: How many ways can you do this?

• You choose exactly 4 toppings from a list of 11 options. Double, triple, quadruple topping all ok.

Let the number of times a topping is chosen as  $x_i \ge 0$  and  $\sum_{i=1}^{11} x_i = 4$ .

This is a distributing 4 identical items (toppings) into 11 distinct bins (topping options).

$$\left\langle \begin{array}{c} 11\\ 4 \end{array} \right\rangle = \left( \begin{array}{c} 11+4-1\\ 4 \end{array} \right)$$

• You choose at most 4 toppings from a list of 11 options. Double, triple, quadruple topping all ok.



# Lattice path

- How many shortest lattice path start at (0,3) and end at (3,0)?
- Lattice = we are on a grid with lattice points
- Analysis: draw a grid

- To reach B from A in a shortest path, we can only move South (S) or East (E).
- Moving from A to B has 6 steps, e.g., EEESSS
- EEESSS is the same as "in 6 boxes, put 3 E and 3 S"
- So the question is the same as "how many ways to put 3 E and 3 S in 6 boxes".

$$\begin{pmatrix} 6\\ 3 \end{pmatrix}$$

#### Lattice path examples

• **Theorem** In a 2D plane, the number of shortest lattice paths from (0,0) to (m,n) is  $\binom{m+n}{m}$ 

• E.g. From 
$$(0,0)$$
 to  $(1,1)$ , there are  $\begin{pmatrix} 1+1\\1 \end{pmatrix} = 2$  shortest lattice paths with step 2.

• E.g. From 
$$(0,0)$$
 to  $(2,2)$ , there are  $\begin{pmatrix} 2+2\\2 \end{pmatrix} = 6$  shortest lattice paths with step 4.



• E.g. From 
$$(2,2)$$
 to  $(3,4)$ , there are  $\begin{pmatrix} 1+2\\1 \end{pmatrix} = 3$  shortest lattice paths with step 3.

**E.g.** Lattice paths from (0,0) to (10,10) that pass through (3,3) but not (5,5)? • Product rule: consider  $\{(0,0) \rightarrow (3,3)\} \times \{(3,3) \rightarrow (10,10) \text{ that do not pass through } (5,5)\}$ •  $\left\{ (0,0) \rightarrow (3,3) \right\}$  has  $\binom{3+3}{2}$  possibilities. •  $\{(3,3) \rightarrow (10,10) \text{ that do not pass through } (5,5)\} = \{(0,0) \rightarrow (7,7) \text{ that do not pass through } (2,2)\}$ • Complement  $\{(0,0) \rightarrow (7,7) \text{ that do not pass through } (2,2)\} = \text{all paths } -\{(0,0) \rightarrow (7,7) \text{ that pass through } (2,2)\}$ • Treat  $\{(0,0) \rightarrow (7,7) \text{ that pass through } (2,2)\}$  using product rule again  $\left|\left\{(0,0) \rightarrow (3,3)\right\} \times \left\{(3,3) \rightarrow (10,10) \text{ that do not pass through } (5,5)\right\}\right|$  $\stackrel{\text{prod. rule}}{=} \left| \left\{ (0,0) \rightarrow (3,3) \right\} \right| \cdot \left| \left\{ (0,0) \rightarrow (7,7) \text{ that do not pass through } (2,2) \right\} \right|$  $\stackrel{\text{complement}}{=} \begin{pmatrix} 3+3\\ 3 \end{pmatrix} \left( \left| \left\{ (0,0) \rightarrow (7,7) \right\} \right| - \left| \left\{ (0,0) \rightarrow (7,7) \text{ that pass through } (2,2) \right\} \right| \right)$  $= \binom{6}{3} \left( \binom{7+7}{7} - \left| \left\{ (0,0) \to (7,7) \text{ that pass through } (2,2) \right\} \right| \right)$ 

$$\begin{array}{ll} \begin{array}{l} \operatorname{prod. rule} & \binom{6}{3} \left( \binom{14}{7} - \left| \left\{ \{(0,0) \to (2,2) \right\} \times \{(2,2) \to (7,7) \} \right\} \right| \right) \end{array} \right| \\ \begin{array}{l} \operatorname{prod. rule} & \binom{6}{3} \left( \binom{14}{7} - \left| \{(0,0) \to (2,2) \right\} \right| \cdot \left| \{(2,2) \to (7,7) \} \right| \right) \\ \end{array} \\ \begin{array}{l} \operatorname{prod. rule} & \binom{6}{3} \left( \binom{14}{7} - \binom{2+2}{2} \binom{5+5}{5} \right) = \binom{6}{3} \left( \binom{14}{7} - \binom{4}{2} \binom{10}{5} \right) \end{array} \right) \end{array}$$

# Lattice path and multinomial coefficient

- Theorem In a 2D plane, the number of lattice paths from (0,0) to (m,n) is  $\binom{m+n}{m}$
- **Theorem** In a 3D cube, the number of lattice paths from (0,0,0) to (m,n,p) is  $\binom{m+n+p}{m,n,p}$



- How many lattice paths from (0, 1, 2) to (5, 5, 5) that pass through (3, 3, 3) but not (1, 2, 3)?
- Generalization In a *n*-dimensional hypercube, the number of lattice paths from (0, 0, ..., 0) to  $(k_1, k_2, ..., k_n)$  is  $\begin{pmatrix} k_1 + ... + k_n \\ k_1, ..., k_n \end{pmatrix}$ .

# Contents

ounting by set theory Inclusion-exclusion principle Enumeration, sum rule & product rule

Counting by binomial coefficient Permutation, factorial and division rule Combination and binomial Binomial expansion Trinomial & multinomial

#### Advanced counting techniques

Counting via bijection Counting via generating function / z-transform Recursion, partial fraction and generating function Pigeonhole principle

Advanced<sup>2</sup> counting techniques Gosper's algorithm Hypergeometric function

# Counting via bijection

- Assume we have a bijection from set X to set Y
- "If counting on X is hard, count on Y instead"
- Counting via bijection is a powerful technique.
- E.g.  $[x^r](1+x)^n$  and # of r-combinations of n-set is a bijection. This is why we can use binomial coefficient to count things.



Bijection: one-to-one (injective) and onto (surjective)

# **Rising number**

• Definition A positive integer is called a *rising number* if its digits form a strictly increasing sequence.

Rising number := 
$$\Big\{ d_1 d_2 \cdots d_n \mid d_i \in \{1, ..., 9\}, \, d_1 < d_2 < \cdots < d_n \Big\}.$$

Number	is rising?		
123	yes		
343	no		
2334	no		
1	yes		

• How many 3-digit rising numbers are there? Let

$$\mathcal{S} \coloneqq \Big\{ d_1 d_2 d_3 \, | \, d_i \in \{1, ..., 9\}, \, d_1 < d_2 < d_3 \Big\}.$$

The question is asking: what is |S|?

#### Counting rising number via enumeration, product rule and sum rule

• Case *d*<sub>1</sub> = 1



Total number of cases: 28 + 21 + 15 + 10 + 6 + 3 + 1 = 84 69 / 118

# Counting rising number by bijection

• Fact 1:  $d_1 < d_2 < d_3$  (rising number)  $\implies d_1 \neq d_2 \neq d_3$  (3 distinct digits)

Fact 2: d<sub>1</sub> ≠ d<sub>2</sub> ≠ d<sub>3</sub> (3 distinct digits) ⇒ there is exactly 1 rising number.
 e.g. for {4,7,1}, the raising number is 147 and there is only one possible raising number.

• Hence there is a bijection between

# of 3-digit rising numbers and # of ways of selecting 3 distinct digits

# of ways of selecting 3 distinct digits from 9 symbols = 9-choose-3 = 
$$\begin{pmatrix} 9 \\ 3 \end{pmatrix} = \frac{9!}{6!3!} = 84.$$

# **Examples of rising numbers**

- E.g. How many 4-digit rising numbers are there that are greater than 5000?
- >5000 means we are limited to  $\{5, 6, 7, 8, 9\}$
- By the bijection argument, the number is 5-choose-4  $\binom{5}{4} = 5$ .

Or you can choose to list them

5678, 5679, 5689, 5789, 6789

- E.g. How many 4-digit rising numbers with  $d_4 = 9$  are there?
- $d_4$  is fixed so we do not need to consider  $d_4$
- This problem is the same as counting how many 3-digit rising numbers with  $d_i \in \{1, 2, ..., 8\}$

# $\binom{8}{3}$ .

- E.g. With alphabet  $\{a, b, ..., z\}$ , how many 5-letter rising strings with  $d_2 = d$  and  $d_3 = f$  are there?
- This problem is the same as counting how many rising strings as "?df??"
- The first "?" has 3 choices:  $\{a, b, c\}$
- The last two "??" have  $\binom{20}{2}$ : choosing  $\{g,h,i,...,z\}$  for 2 box
- So the total number is  $3\binom{20}{2} = 570$

# Example of counting via bijection

- (Bona 2023) A city has recently built ten intersections. Some of these will get traffic lights, and some of those that get traffic lights will also get a gas station. In how many different ways can this happen?
- Let  $\{A, B, C\}$  be three choices. Each of the intersections can be in one of three states:
- A No traffic light.
- *B* Traffic light only.
- C Traffic light and gas station.
- As the two problem has a bijection. So we count by bijection.
- The total number of different ways to assign traffic lights and gas stations is calculated as 3<sup>10</sup>.
Bijection: composition of number, bit strings, tildes



Bijection between 3-bit, tildes and compositions of 4

- Composition of a integer means writing this number as a sum of other distinct integers.
- E.g. 5 = 1 + 4 is a composition, 5 = 1 + 1 + 1 + 2 is also a composition.

#### Twelvefold way of balls and boxes

f-class	any $f$ (no rules on placement)	Injective $f$ (no multi-packs allowed)	Surjective $f$ (no empty box allowed)
Distant <i>f</i>	How many ways can you place $n$ marked balls into $x$ marked boxes, with no other rules on placement? $x^n$	How many ways can you place $n$ marked balls into $x$ marked boxes, with no multipacks allowed? $x(x-1)(x-n+1)=x^{\underline{n}}$	How many ways can you place $n$ marked balls into $x$ marked boxes, with no empty boxes allowed? out of scope
$S_n$ orbit $f \circ S_n$	How many ways can you place $n$ plain balls into $x$ marked boxes, with no other rules on placement? $\left\langle \begin{array}{c} x \\ n \end{array} \right\rangle = \binom{x+n-1}{n}$	How many ways can you place $n$ plain balls into $x$ marked boxes, with no multi-packs allowed? $\binom{x}{n}$	How many ways can you place $n$ plain balls into $x$ marked boxes, with no empty boxes allowed? $\binom{n-1}{n-x}$
$egin{array}{ccc} S_x & { m orbit} \ S_x \circ f \end{array}$	How many ways can you place $n$ marked balls into $x$ plain boxes, with no other rules on placement? out of scope	How many ways can you place $n$ marked balls into $x$ plain boxes, with no multi- packs allowed? out of scope	How many ways can you place $n$ marked balls into $x$ plain boxes, with no empty boxes allowed? out of scope
$S_n, S_x$ orbit $S_x \circ f \circ S_n$	How many ways can you place $n$ plain balls into $x$ plain boxes, with no other rules on placement? out of scope	How many ways can you place $n$ plain balls into $x$ plain boxes, with no multi-packs al- lowed? out of scope	How many ways can you place $n$ plain balls into $x$ plain boxes, with no empty boxes allowed? out of scope

# Application of bijection: cardinality of infinite set

Not in exam

$$|\mathcal{S}| \begin{cases} = 0 & \text{ if } \mathcal{S} = \emptyset \\ < \infty & \text{ if } \mathcal{S} \text{ is finite} \end{cases}$$

- Definition S is countable if either S is finite, or S is infinite with an *bijection*<sup>2</sup> with  $\mathbb{N}$
- $|\mathbb{N}| \coloneqq \aleph_0$  (Aleph null), it means "countably infinite"

think of it as  $\infty$ 

- A set S is countable if  $|S| \leq \aleph_0$ , a set is uncountable if  $|S| > \aleph_0$
- Cantor's diagonal argument for bijection of infinite set
- Let  $\mathbb{N}$  be the set of natural number
- Let  $\mathbb{E}$  be the set of positive even integers
- Which set has more elements?

<sup>2</sup>one-to-one (injective) onto (surjective)

# Algebraic methods in enumerative combinatorics

- Question: is there a more systemic way to counting? Answer: yes
- Recall the very beginning:

		single term	n double term	n triple term	quadruple term
A		1			
$ A \cup B $		2	1		
$ A \cup B \cup C $		3	3	1	
$ A\cup B\cup C$	$\cup D $	4	6	4	1
	powe	r-1 term 🛛	oower-2 term	power-3 term	power-4 term
$(1+x)^1$		1			
$(1+x)^2$		2	1		
$(1+x)^3$		3	3	1	
$(1+x)^4$		4	6	4	1

we can count things using the coefficients of polynomials

(: they have the same algebra)

• We can count things using the coefficients of polynomials because they have a bijection.

# Counting via polynomial coefficient, example

- E.g. For a, b, c in  $\mathbb{Z}_+$ , if  $2 \le a \le 5, 3 \le b \le 6$  and  $4 \le c \le 7$ , how many solutions to a + b + c = 17?
- It is the coefficient of  $x^{17}$  in the polynomial

$$(x^{2} + x^{3} + x^{4} + x^{5})(x^{3} + x^{4} + x^{5} + x^{6})(x^{4} + x^{5} + x^{6} + x^{7})$$

- E.g. You distribute 8 identical cookies among 3 children. Each child can receive at least 2 cookies and no more than 4 cookies. How many ways can you distribute?
- It is the coefficient of  $x^8$  in the polynomial

$$(x^{2} + x^{3} + x^{4})(x^{2} + x^{3} + x^{4})(x^{2} + x^{3} + x^{4}).$$

• How come? Because there is a bijection between the problem and the polynomial coefficient.

#### **Restricted Partition vs Partition**

- E.g. For a, b, c in  $\mathbb{Z}_+$ , if  $2 \le a \le 5, 3 \le b \le 6$  and  $4 \le c \le 7$ , how many solutions to a + b + c = 17?
  - Restriction 1: we fix three numbers sum to 17
  - Restriction 2: we restrict the possible values of the numbers
- Def (Partition)

P(n) represents the number of possible partitions of a non-negative integer n

- No restriction on how many numbers to sum to n
- No restriction on possible values of the numbers (other than in  $\mathbb{N}$ )
- Discovering a closed-form expression for the partition function P(n) is an open problem

#### **Closed-form expression for the partition function** P(n) **is open problem**



1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, ...

# Six-sided die

- Die = singular form of dice in old-fashioned English
- All the possible outcome of 2d6 (toss a six-sided die twice)



• Consider a polynomial  $x^1 + x^2 + x^3 + x^4 + x^5 + x^6$ All the terms in  $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$ 

### Defective six-sided die

• Suppose the die is defective that output 1 is impossible. All the possible outcome of 2d6



• Consider a polynomial  $0x^1 + x^2 + x^3 + x^4 + x^5 + x^6$ All the terms in  $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$ , where we ignore  $x^1$ 

> > $1x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12}$

#### Distribute cookies to two children

• How many ways to distribute 8 cookies to two children with at least 1 cookies and no more than 6 cookies

 $x^{5}x^{1}$  $x^3$  $x^7$  $x^1x^1$  $x^{2}x^{1}$  $x^{3}x^{1}$  $x^4x^1$  $x^{6}x^{1}$  $x^2$  $x^4$  $x^5$  $x^6$  $x^2x^2$  $x^{3}x^{2}$  $x^3$  $x^4$  $x^{1}x^{2}$  $x^{4}x^{2}$  $x^{5}x^{2}$  $x^{6}x^{2}$  $x^5$  $x^6$  $x^7$  $x^8$  $x^{1}x^{3}$  $x^{2}x^{3}$  $x^{3}x^{3}$  $x^4x^3$  $x^{5}x^{3}$  $x^{6}x^{3}$  $\frac{x^4}{x^5}$  $x^5$  $x^6$  $x^7$  $x^8$  $x^9$  $x^6$  $x^7$  $x^1 x^4$  $x^{2}x^{4}$  $x^3x^4$  $x^4x^4$  $x^{5}x^{4}$  $x^{6}x^{4}$  $x^8$  $x^9$  $x^{10}$  $x^6$  $x^7$  $x^8$  $x^{1}x^{5}$  $x^{2}x^{5}$  $x^{3}x^{5}$  $x^{4}x^{5}$  $x^{5}x^{5}$  $x^{6}x^{5}$  $x^9$  $x^{10}$  $x^{11}$  $\frac{\pi}{r^7}$  $x^8$  $x^{2}x^{6}$  $x^9$  $x^{1}x^{6}$  $x^{3}x^{6}$  $x^{4}x^{6}$  $x^{5}x^{6}$  $x^{6}x^{6}$  $x^{10}$  $x^{11}$  $r^{12}$ 

• How many ways to distribute 7 cookies to two children with at least 2 cookies and no more than 5 cookies

 $x^7$  $x^1x^1$  $x^{2}x^{1}$  $x^{3}x^{1}$  $x^4x^1$  $x^{5}x^{1}$  $x^{6}x^{1}$  $x^2$  $x^3$  $x^4$  $x^5$  $x^{1}x^{2}$  $x^2x^2$  $x^{3}x^{2}$  $x^{4}x^{2}$  $x^{6}x^{2}$  $x^3$  $x^4$  $x^6$  $x^7$  $x^{5}x^{2}$  $x^5$  $x^8$  $egin{array}{ccc} x^4 & x^5 \ x^5 & x^6 \end{array}$  $x^6$  $x^7$  $x^{1}x^{3}$  $x^{2}x^{3}$  $x^{3}x^{3}$  $x^4x^3$  $x^{6}x^{3}$  $x^7$  $x^{5}x^{3}$  $x^8$  $x^9$  $x^6 x^4 \implies$  $x^8$  $x^{1}x^{4}$  $x^2x^4$  $x^3x^4$  $x^4x^4 \quad x^5x^4$  $x^9$  $x^{10}$  $x^5x^5$   $x^6x^5$   $x^6$   $x^7$   $x^8$  $x^5x^6$   $x^6x^6$   $x^7$   $x^8$   $x^9$  $x^{1}x^{5}$  $x^2 x^5 = x^3 x^5$  $x^{4}x^{5}$  $x^9$  $x^{10}$  $x^{11}$  $x^{1}x^{6}$  $r^2 r^6$  $x^{3}x^{6}$  $x^{4}x^{6}$  $r^{10}$  $r^{11}$  $x^{12}$ 

 $\iff (x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + x^5)$ 

#### Number of integer solutions to an equation

- E.g. How many solution to the equation a + b + c = 17 if a, b, c are integers and  $2 \le a \le 5, 3 \le b \le 6$  and  $4 \le c \le 7$ ?
- The solution is the coefficient of  $x^{17}$  in the polynomial

$$(x^{2} + x^{3} + x^{4} + x^{5})(x^{3} + x^{4} + x^{5} + x^{6})(x^{4} + x^{5} + x^{6} + x^{7})$$

(which is 3)

• The same as plotting all the (i, j, k) coordinate of the outcomes and find the total number of terms that has power 17.



Power series  $G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$ 

- E.g. Consider a 1-set {e<sub>1</sub>}
  1 way to select zero elements
  1 way to select one element
  0 way to select more than 1 element
- E.g. Consider a 2-set { $e_1, e_2$ } 1 way to select zero elements

2 way to select one element

 $1 \ \mbox{way to select two elements}$  (we take combination)

0 way to select more than 2 elements

• 
$$1 + 2x + x^2 = (1 + x) \cdot (1 + x)$$

$$1x^{0} + 1x^{1} + 0x^{2} + 0x^{3} + \dots = 1 + x$$

$$1x^{0} + 2x^{1} + x^{2} + 0x^{3} + \dots = 1 + 2x + x^{2}$$

$$\underbrace{(1+x)}_{\text{select from } \{e_1\}} \underbrace{(1+x)}_{\text{product rule select from } \{e_2\}} \underbrace{(1+x)}_{\text{select from } \{e_2\}} = \underbrace{(1+x)^2}_{\text{select from } \{e_1,e_2\}} = 1 + 2x + x^2$$

• In fact these are binomial expansion of  $(1+x)^k$ 

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#### Sequence and generating function

• E.g. Given a sequence 
$$\underbrace{\{1,1,1,1,1,1\}}_{six \ 1}$$
, its generating function is  
 $1 + x + x^2 + x^3 + x^4 + x^5$ .

Note that we have three representations of the same thing

$$\underbrace{f_n = \begin{cases} 1 & n \in \{0, 1, 2, 3, 4, 5\} \\ 0 & \text{other wise} \end{cases}}_{\text{definition form}} \iff \underbrace{\{1, 1, 1, 1, 1\}}_{\text{a list}} \iff \underbrace{1 + x + x^2 + x^3 + x^4 + x^5}_{\text{generating function}}$$

- E.g. For sequence  $\{0, 1, 2, 3\}$ , its generating function is  $x + 2x^2 + 3x^3$ .
- E.g. For sequence  $\{0, 1, 0, 1, 0, 1, 0, 1, ...\}$ , its generating function is  $x + x^3 + x^5 + x^7 + ...$

# Example of product rule and generating function

- E.g. A word must be started with a prefix, followed by a vowels, and ends in a suffix. How many words can prefixes =  $\{qu, s, t\}$ you build from vowels =  $\{a, i, oi\}$ ? suffixes =  $\{d, ff, ck\}$
- Solution by enumerating the  $3^3 = 27$  outcomes

```
sad sid tad tid
quad quid sack saff sick siff soid tack taff tick tiff toid
quack quaff quick quiff quoid soick soiff toick toiff quoick quoiff
```

There are 4 length-3 words, 12 length-4 words, 9 length-5 words, and 2 length-6 words.

• How to count number of length-5 words without enumeration? We construct a product of generating function

$$\begin{aligned} (qu+s+t)(a+i+oi)(d+ff+ck) &= (xx+x+x)(x+x+xx)(x+xx+xx) \\ &= (2x+x^2)(2x+x^2)(x+2x^2) \\ &= 4x^3+12x^4+9x^5+2x^6. \end{aligned}$$

(we don't care about the exact letter, we replace each letter by x) (Why qu is xx: the xx represents "select or not select 2 letters")

• Coefficient of  $x^5 = \#$  of length-5 words = 9.

# Product rule and sum rule, in generating function

• Let  $\begin{cases} A(x) \text{ be the gen. func. for selecting items from set } \mathcal{A} \\ B(x) \text{ be the gen. func. for selecting items from set } \mathcal{B} \end{cases}$ 

• E.g. Rolling a six-sided die four times. By product rule, the GF is

 $C(x) = (x^{1} + x^{2} + x^{3} + x^{4} + x^{5} + x^{6})^{4}$ 

The number of ways that the result add up to 20 is the coefficient of  $x^{20}$ . Then how do you find the coefficient: multinomial coefficient.

- Product rule (GF ver.) The GF for the set  $C := A \times B$  is C(x) = A(x)B(x).
- Sum rule of disjoint set (GF ver.)
   If A and B are disjoint, the GF for the set C = A ∪ B is C(x) = A(x) + B(x)
- Is there a GF for inclusion-exclusion principle? Yes C(x) = A(x) + B(x) A(x)B(x).

# Generatingfunctionology / Algebraic Combinatorics

- **Product rule** A(x)B(x).
- Sum rule of disjoint set A(x) + B(x)
- Inclusion-exclusion principle A(x) + B(x) A(x)B(x).
- Division rule  $\frac{A(x)}{B(x)}$ .
- Permutation

• Taylor series  $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ 

• Exponential generating function  $\sum_{k=0}^{\infty} a_n \frac{x^k}{k!}$ 

- Generalized permutation  $A(x) = \prod_{i=1}^n \frac{1}{1-x^{k_i}}$
- Generalized combination, Sterling functions, raising factorial, ...
- $\frac{d}{dx}$  has a combinatorics meaning: counting structures with ONE distinguished Element
- $\frac{d^2}{dx^2}$  has a combinatorics meaning: counting structures with TWO distinguished Element
- ∫ dx has a combinatorics meaning: counting structures with accumulation
- log has a combinatorics meaning: counting connected structures (cycle)
- Now you should view math from a new perspective.
- there is something behind

#### **Coefficient in infinite series**

On 
$$1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$
 for  $|x| < 1$ , take differentiation  $n-1$  times gives  
$$\sum_{k=n-1}^{\infty} k(k-1)(k-2)\cdots(k-n+2)x^{k-n+1} = \frac{(n-1)!}{(1-x)^n}$$

Let r = k - n + 1 then k = r + n - 1

$$\sum_{r+n-1=n-1}^{\infty} (r+n-1)(r+n-2)(r+n-3)\cdots(r+1)x^r = \frac{(n-1)!}{(1-x)^n}$$

$$\iff \sum_{r=0}^{\infty} (r+n-1)(r+n-2)(r+n-3)\cdots(r+1)\frac{r!}{r!}x^r = \frac{(n-1)!}{(1-x)^n}$$

$$\iff \sum_{r=0}^{\infty} \frac{(r+n-1)!}{r!}x^r = \frac{(n-1)!}{(1-x)^n}$$

$$\iff \sum_{r=0}^{\infty} \frac{(r+n-1)!}{(n-1)!r!}x^r = \frac{1}{(1-x)^n} = (1+x+x^2+\cdots)^n$$

$$(1 + x + x^{2} + \dots)^{n} = \sum_{r=0}^{\infty} {\binom{n+r-1}{r}} x^{r} = \sum_{r=0}^{\infty} {\binom{n}{r}} x^{r}$$

**E.g.**  $[x^{22}](1+x+x^2+\cdots)^1 = \binom{1+22-1}{22} = 1, \quad [x^{22}](1+x+x^2+\cdots)^8 = \binom{8+22-1}{22} = \binom{29}{22} = \frac{29!}{22!7!} = \binom{29}{22} = \binom{29}{22!7!} = \binom{29}{22} = \binom{29}{22!7!} = \binom{29}{2!7!} = \binom{29}{2!7!7!} = \binom{29}{2!7!7!} = \binom{29}{2!7!7!$ 

Example

$$\left| (1+x+x^2+\cdots)^n = \sum_{r=0}^{\infty} \left\langle {n \atop r} \right\rangle x^r \right|$$

$$\begin{aligned} \mathbf{E.g.} \ [x^{22}](x+x^2+\cdots)^3 & \text{ is the same as } [x^{19}](1+x^2+\cdots)^3, \text{ which is } \left\langle \begin{array}{c} 3\\19 \right\rangle = \binom{3+19-1}{19} = \binom{21}{19} = 210. \text{ Or,} \\ (x+x^2+\cdots)^3 & = (-1+1+x+x^2+\cdots)^3 \\ & = \left(-1+(1+x+x^2+\cdots)\right)^3 \\ & = \sum_{k=0}^3 \binom{3}{k}(-1)^{3-k}(1+x+x^2+\cdots)^k \\ & = \binom{3}{0}(-1)^{3-0}(1+x+x^2+\cdots)^0 + \binom{3}{1}(-1)^{3-1}(1+x+x^2+\cdots)^1 \\ & +\binom{3}{2}(-1)^{3-2}(1+x+x^2+\cdots)^2 + \binom{3}{3}(-1)^{3-3}(1+x+x^2+\cdots)^3 \\ & = -1+3(1+x+x^2+\cdots)-3(1+x+x^2+\cdots)^2 + (1+x+x^2+\cdots)^3 \end{aligned}$$

The coefficient is thus

$$0 + 3\binom{1+22-1}{22} - 3\binom{2+22-1}{22} + \binom{3+22-1}{22} = 3 - 3(23) + 276 = 210.$$

# Using generating function to solve difficult counting problem ... (1/2)

- Solve it using generating function
  - We can select apples in way of  $\{1, 0, 1, 0, ...\}$   $\begin{cases}
    1 way for zero apple \\
    0 way for one apple \\
    1 way for two apples \\
    0 way for three apples \\
    so on
    \end{cases}$

The generating function for apple is  $A(x) = 1 + x^2 + x^4 + x^6 + \dots$ • The generating function for Banana,  $B(x) = 1 + x^5 + x^{10} + x^{15} + \dots$ 

- The generating function for orange,  $O(x) = 1 + x^1 + x^2 + x^3 + x^4$  The generating function for pear,  $P(x) = 1 + x^1$
- By product rule (GF ver.), the generating function of slecting the fruits is  $A(x)B(x)O(x)P(x) = \left(1 + x^2 + x^4 + x^6 + \dots\right)\left(1 + x^5 + x^{10} + x^{15} + \dots\right)\left(1 + x^1 + x^2 + x^3 + x^4\right)\left(1 + x^1\right)$ Let's call A(x)B(x)O(x)P(x) as Fruit(x). 91 / 118

**E.g.** How many ways to fill a bag with n fruits if at most four oranges at most one pear

# Using generating function to solve difficult counting problem ... (2/2)

$$\begin{aligned} \text{Fruit}(x) &= \left(1 + x^2 + x^4 + \dots\right) \left(1 + x^5 + x^{10} + \dots\right) \left(1 + x^1 + x^2 + x^3 + x^4\right) \left(1 + x^1\right) \\ &= \left(1 + x^2 + x^4 + \dots\right) \left(1 + x^5 + x^{10} + \dots\right) \left(\frac{1 - x^5}{1 - x}\right) \left(1 + x^1\right) \\ &= \left(\frac{1}{1 - x^2}\right) \left(\frac{1}{1 - x^5}\right) \left(\frac{1 - x^5}{1 - x}\right) \left(1 + x^1\right) \\ &= \left(\frac{1}{(1 - x)(1 + x^4)}\right) \frac{1}{1 - x^5} \frac{1 - x^5}{1 - x} (1 + x^4) \\ &= \frac{1}{(1 - x)^2} \\ &= \frac{d}{dx} \frac{1}{1 - x} \\ &= \frac{1}{dx} \left(1 + x + x^2 + x^3 + \dots\right) \\ &= 1 + 2x^1 + 3x^2 + 4x^3 + \dots \\ &= (n + 1)x^n \big|_{n=0} + (n + 1)x^n \big|_{n=1} + (n + 1)x^n \big|_{n=2} + (n + 1)x^n \big|_{n=3} + \dots \end{aligned}$$
So there are  $(n + 1)$  ways to select *n* fruits. 92/118 
$$\end{aligned}$$

#### Multiplicative inverse of power series

• **Definition** A polynomial B(x) is a inverse of A(x) if B(x)A(x) = 1.

• E.g. 
$$1 - x$$
 is the inverse of  $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$   
 $(1 - x) \sum_{k=0}^{\infty} x^k = (1 - x) (1 + x + x^2 + x^3 + \cdots) = \frac{1 + x + x^2 + x^3 + \cdots}{-x - x^2 - x^3 - \cdots} = 1.$   
In fact, by geometric series,  $1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$ .

• Generalization By geometric sum, we have

$$1 + rx + r^2x^2 + \dots = \frac{1}{1 - rx}$$

Illustration using multiplicative inverse

$$(1-rx)\left(1+rx+r^{2}x^{2}+r^{3}x^{3}+\cdots\right) = \frac{1+rx+r^{2}x^{2}+r^{3}x^{3}+\cdots}{-rx-r^{2}x^{2}-r^{3}x^{3}-\cdots} = 1.$$

# **Basic formulas**

• Binomial expansion

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

• Finite Geometric series

$$1 + x + x^{2} + \dots + x^{n} = \frac{1 - x^{n+1}}{1 - x}$$

• Power of infinite Geometric series

$$(1 + x + x^{2} + \dots)^{n} = \left(\frac{1}{1 - x}\right)^{n} = \sum_{r=0}^{\infty} {n + r - 1 \choose r} x^{r} = \sum_{r=0}^{\infty} \left\langle \frac{n}{r} \right\rangle x^{r}$$

• Differentiation

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

• Trinomial (counting two type of objects)

$$(1+x+y)^n = \sum_{k=0}^n \binom{n}{i,j} x^i y^j$$

# What's the big deal of generating function?

#### Not in exam

- A more systematic approach to counting: you count by using polynomial coefficient.
- However, you need a lots of mathematics to use generating function
  - Finite sum and infinite sum
  - Differentiation and integration
  - Partial fraction
- Generating function can be used to solve recursion problem
- Such as finding the closed-form solution of Fibonacci number as  $f_n = \frac{(-1)^{n+1}}{\sqrt{5}} \left( \varphi_-^{n+1} \varphi_+^{n+1} \right)$ , where

$$\varphi_{\pm} = rac{-1 \pm \sqrt{5}}{2}$$
 is known as golden ratio.

- In fact generating function is called *z*-transform in digital signal processing.
- It is called z because it uses complex number.
- In fact, given a generating function, we can get back the series via inverse Z-transform via

$$x[n] = \frac{1}{2\pi\sqrt{-1}} \oint_C X(z) z^{n-1} \mathrm{d}z,$$

in which you need the knowledge of complex number and complex integration.

Techniques for generating function : partial fraction • E.g.  $\frac{1}{(1-x)(1-2x)} = \frac{x}{1-x} + \frac{x}{1-2x}$  $\frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} \iff 1 = A(1-2x) + B(1-x).$ Put x = 1 gives A = -1. Put  $x = \frac{1}{2}$  gives B = 2, so  $\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}$ . • E.g.  $\frac{17x-53}{x^2-2x-15} = \frac{?}{x-5} + \frac{?}{x-5}$  $\frac{17x-53}{x^2-2x-15} = \frac{17x-53}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \iff 17x-53 = A(x+3) + B(x-5).$ Put x = 5 gives A = 4 and put x = -3 gives B = 13. Hence  $\frac{17x - 53}{x^2 - 2x - 15} = \frac{4}{x - 5} + \frac{13}{x - 2}$ • **E.g.**  $\frac{10+35}{(x+4)^2} = \frac{10}{x+4} + \frac{-5}{(x+4)^2}$  $\frac{10x+35}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2} \iff 10x+35 = A(x+4) + B.$ Put x = -4 gives B = -5 and put x = 0 with B = -5 gives A = 10.

#### Solving recursion via generating function, example $1 \dots (1/2)$

- Consider a function defined recursively as  $f_n = \begin{cases} 2 & n = 0 \\ 3f_{n-1} 1 & n \ge 1 \end{cases}$ . What is the general formula of  $f_n$ ?
- Enumerating the solution for multiple n and then guessing the solution is not very helpful.

Let 
$$F(x) = \sum_{\substack{n=0 \ F(x)}}^{\infty} f_n x^n$$
. Now  $f_n = 3f_{n-1} - 1 \iff f_n - 3f_{n-1} = -1$ .  
 $F(x) = f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \cdots$  by definition  
 $-3xF(x) = -3f_0 x^1 - 3f_1 x^2 - 3f_2 x^3 - \cdots$  multiply  $-3x$   
 $(1 - 3x)F(x) = f_0 + (f_1 - 3f_0)x^1 + (f_2 - 3f_1)x^2 + (f_3 - 3f_2)x^3 + \cdots$  add the two equations  
 $= 2 - x^1 - x^2 - x^3 - \cdots$  by  
 $= 2 + (1 - 1) - x^1 - x^2 - x^3 - \cdots$   
 $= 3 - (1 + x^1 + x^2 + x^3 + \cdots)$   
 $= 3 - \frac{1}{1 - x}$  geometric series

• Now we have

$$(1-3x)F(x) = 3 - \frac{1}{1-x} \qquad \Longleftrightarrow \qquad F(x) = \frac{3}{1-3x} - \frac{1}{(1-x)(1-3x)}$$
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# Solving recursion via generating function, example $1 \dots (2/2)$

• Apply partial fraction

$$\frac{1}{(1-x)(1-3x)} = \frac{\frac{1}{1-3}}{1-x} + \frac{\frac{1}{1\frac{1}{3}}}{1-3x} = \frac{-\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}$$
• Then  $(1-3x)F(x) = 3 - \frac{1}{1-x}$  gives  

$$F(x) = \frac{3}{1-3x} + \frac{\frac{1}{2}}{1-x} - \frac{\frac{3}{2}}{1-3x}$$

$$= \frac{\frac{3}{2}}{1-3x} + \frac{\frac{1}{2}}{1-x}$$

$$= \frac{3}{2}\frac{1}{1-3x} + \frac{1}{2}\frac{1}{1-x}$$

$$= \frac{3}{2}\left(1+3x+(3x)^2+(3x)^3+\cdots\right) + \frac{1}{2}\left(1+x+x^2+x^3+\cdots\right)$$

• Therefore

$$f_n = \frac{3}{2} \cdot 3^n + \frac{1}{2} \cdot 1^n = \frac{3}{2} \cdot 3^n + \frac{1}{2}$$

You can try with  $n=\{0,1,2,3\}$ 

# Solving recursion via generating function, example 2

• Find the closed-form expression of the recursion 
$$f_n = \begin{cases} b & n = 0 \\ f_{n-1} + c & n \ge 1 \end{cases}$$

• Let 
$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$
. Now  $f_n = f_{n-1} + c \iff f_n - f_{n-1} = c$ .  

$$F(x) = f_0 + f_1 x^1 + f_2 x^2 + \cdots \qquad \text{by definition} \\ -\frac{1}{x} F(x) = -f_0 \frac{1}{x} - f_1 - f_2 x^1 - f_3 x^2 - \cdots \qquad \text{multiply } -\frac{1}{x} \\ (1 - \frac{1}{x}) F(x) = -f_0 \frac{1}{x} + (f_0 - f_1) + (f_1 - f_2) x^1 + (f_2 - f_3) x^2 + \cdots \\ (1 - \frac{1}{x}) F(x) = -f_0 \frac{1}{x} - (f_1 - f_0) - (f_2 - f_1) x^1 - (f_3 - f_2) x^2 - \cdots \\ = -b/x - c - cx^1 - cx^2 - \cdots \qquad \text{by and } f_0 = b \\ multiply - x \\ (1 - x) F(x) = b + c(x^1 + x^2 + x^3 + \cdots) \\ = b - c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - (c + c(1 + x^1 + x^2 + x^3 + \cdots) \\ = b - (c + c(1 + x^1 + x^2 + x^3 + \cdots)) \\ = (b - c)(1 + x + x^2 + x) + c(1 + 2x + 3x^2 + \cdots) \\ = (b - c)(1 + x + x^2 + \cdots) + c(1 + 2x + 3x^2 + \cdots) \\ = b + (b + c)x + (b + 2c)x^2 + (b + 3c)x^3 + \cdots \\ f_n = b + nc$$

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# Solving recursion via generating function, example 3 Find the closed-form expression of the recursion $f_n = \begin{cases} 0 & n=0\\ 1 & n=1\\ f_{n-2}+1 & n \ge 2 \end{cases}$ $f_n - f_{n-2}$

$$n = 1 \quad \Longleftrightarrow \quad f_n - f_{n-2} =$$
  
 $n \ge 2$ 

$$F(x) = f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \cdots$$
  
-x<sup>2</sup> F(x) = -f\_0 x^2 - f\_1 x^3 - \cdots  
(1 - x<sup>2</sup>) F(x) = f\_0 + f\_1 x + (f\_2 - f\_0) x^2 + (f\_3 - f\_1) x^3 + \cdots  
= x + x<sup>2</sup> + x<sup>3</sup> + ...

by definition  
multiply 
$$-x^2$$
  
add the two equations  
by and  $f_0 = 0, f_1 = 1$   
geo. series,  $\frac{1}{(1+x)(1-x)}$ 

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action

$$\begin{split} F(x) &= \frac{x}{(1+x)(1-x)^2} & \text{geo. serie} \\ &= \frac{-1}{4} \frac{1}{1+x} + \frac{-1}{4} \frac{1}{1-x} + \frac{1}{2} \frac{1}{(1-x)^2} & \text{partial fra} \\ &= \frac{-1}{4} \frac{1}{1-(-x)} + \frac{-1}{4} \frac{1}{1-x} + \frac{1}{2} \frac{d}{dx} \frac{1}{1-x} \\ &= \frac{-1}{4} \left(1-x+x^2-\cdots\right) + \frac{-1}{4} \left(1+x+x^2+\cdots\right) + \frac{1}{2} \frac{d}{dx} \left(1+x+x^2+\cdots\right) \\ &= \frac{-1}{4} \left(1-x+x^2-\cdots\right) + \frac{-1}{4} \left(1+x+x^2+\cdots\right) + \frac{1}{2} \left(1+2x+3x^2+\cdots\right) \\ &= \left(\frac{-1}{4} + \frac{-1}{4} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{-1}{4} + \frac{2}{2}\right) x + \left(\frac{-1}{4} + \frac{-1}{4} + \frac{3}{2}\right) x^2 + \cdots \\ &= \frac{(-1)^{0+1} + (-1) + 2(0+1)}{4} + \frac{(-1)^{1+1} + (-1) + 2(1+1)}{4} x + \frac{(-1)^{2+1} + (-1) + 2(2+1)}{4} x^2 + \cdots \\ f_n &= \frac{(-1)^{n+1} + (-1) + 2(n+1)}{4} = \frac{2n + 1 + (-1)^{n+1}}{4} \end{split}$$

# Solving recursion via generating function, E.g.4 $f_n = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 3f_{n-1} - 2f_{n-2} & n \ge 2 \end{cases}$

Let  $F(x) = \sum_{n=1}^{\infty} f_n x^n$ . By  $f_n = 3f_{n-1} - 2f_{n-2} \iff f_n - 3f_{n-1} + 2f_{n-2} = 0$ , we have  $F(x) = f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \cdots$  $-3xF(x) = -3f_0x^1 - 3f_1x^2 - 3f_2x^3 - \cdots$  $2x^2F(x) = +2f_0x^2 + 2f_1x^3 + \cdots$  $(1 - 3x + 2x^2)F(x) = f_0 + (f_1 - 3f_0)x + (f_2 - 3f_1 + 2f_0)x^2 + (f_3 - 3f_2 + 2f_1)x^3 + \cdots$  $= 1 + (3 - 3)x + 0x^2 + 0x^3 + \cdots$  $F(x) = \frac{1}{2x^2 - 3x + 1}$  $= \frac{1}{(1-2x)(1-x)}$  $= \frac{2}{1-2\pi} + \frac{-1}{1-\pi}$  $= 2\left(1 + 2x + 2^{2}x^{2} + 2^{3}x^{3} + \cdots\right) - \left(1 + x + x^{2} + \cdots\right)$  $f_n = 2^{n+1} - 1$ 

Verify with  $f_n = 1, 3, 7, 15, 31, 63, \dots$ 

# **Techniques in generating function**

#### • Steps

- Let  $F(x) = f_0 + f_1 x^1 + f_2 x^2 + f_3 x^3 + \cdots$
- Based on the recursion definition, multiply F(x) by a polynomial in x
- Work on the algebra to get a power series
- Familiar with common power series
- $1 + r + r^2 + \dots + r^{n-1} = \frac{1 r^n}{1 r}$   $1 + x + x^2 + \dots = \frac{1}{1 x}$
- $1 + 2x + 3x^2 + \dots = \frac{1}{(1 x)^2} = \frac{d}{dx} \frac{1}{1 x}$
- Familiar with partial fraction tricks, such has

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{x-a} + \frac{\frac{1}{b-a}}{x-b}$$

Chapter 7 in Discrete Mathematics and Its application  $\rightarrow$ 

G(x)	$a_k$
$(1+x)^n = \sum_{k=0}^n C(n,k)x^k$	C(n, k)
$= 1 + C(n, 1)x + C(n, 2)x^{2} + \dots + x^{n}$	
$+ax)^{n} = \sum_{k=0}^{n} C(n,k)a^{k}x^{k}$	$C(n,k)a^k$
$= 1 + C(n, 1)ax + C(n, 2)a^{2}x^{2} + \dots + a^{n}x^{n}$	
$(1 + x^r)^n = \sum_{k=0}^n C(n, k) x^{rk}$	$C(n, k/r)$ if $r \mid k$ ; 0 otherwise
$= 1 + C(n, 1)x^{r} + C(n, 2)x^{2r} + \dots + x^{rn}$	
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \dots + x^n$	1 if $k \leq n$ ; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2 x^2 + \cdots$	$a^k$
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$ ; 0 otherwise
$\frac{1}{1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	k + 1
$\frac{1}{1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ = 1 + C(n,1)x + C(n+1,2)x^2 +	C(n + k - 1, k) = C(n + k - 1, n - 1)
$\frac{1}{1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k)$ = $(-1)^k C(n+k-1,n-1)$
$\frac{1}{1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$ $= 1 + C(n-1)ax + C(n+1-2)a^2 x^2 + \dots$	$C(n+k-1,k)a^k$ = $C(n+k-1,n-1)a^k$
$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$	1/k!
$(k+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$ 1

# Advanced generating function

Not in exam

- Hypergeometric function: generating function "on steroid"
- Fibonacci sequence  $f_n = \{0, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$

$$f_n = \frac{(-1)^{n+1}}{\sqrt{5}} \left( \varphi_-^{n+1} - \varphi_+^{n+1} \right), \qquad \varphi_{\pm} = \frac{-1 \pm \sqrt{5}}{2} \text{ golden ratio}$$

 $\bullet$  Rook polynomial  $\iff$  ways of nonattacking rooks in chessboard



- What is the number of ways to place 8 rooks in a 8-by-8 chess board that none of the rook are attacking?
- What about bishop? Rotate the chessboard 45° gives you a rook!
- Queen = Rook + bishop.

# **Pigeonhole principle**

• Put n balls inside m boxes. At least one box has  $\lceil \frac{n}{m} \rceil$  elements.

• E.g. three balls in two boxes, (all balls must go into a box)

$$\begin{array}{c|c|c|c|c|c|c|c|c|} & \operatorname{Box-1} & \operatorname{Box-1} & \operatorname{Box-1} & \operatorname{has} & \lceil \frac{3}{2} \rceil = 2 \text{ balls.} \end{array}$$

- Pigeonhole principle
- If you put n pigeons into m pigeonholes and n > m, then at least 1 pigeonhole must contains more than 1 pigeons. To be exact, that pigeonhole contains ⊥(n-1)/m ⊥ + 1 pigeons.
- If you put kn + 1 pigeons into n pigeonholes and  $k \in \mathbb{Z}_+$ , then at least 1 pigeonhole must contain more than k + 1 pigeons.
- E.g. Suppose there are 26 students and 7 cars to transport them. Then at least 1 car have 4 or more passengers.

 $\lfloor (26-1)/7 \rfloor + 1 = 3 + 1 = 4.$ Distribute as evenly as possible = 3

$$\left\lfloor \frac{26}{7} \right\rfloor = 3$$
 students per car (ignoring the remainder)

The -1 in 26 - 1 is used to account for the worst-case scenario in the application of the Pigeonhole. 104 / 118

# Pigeonhole principle and function mapping

- Suppose we are mapping n elements from the domain to a range via a function f
- Denote |n| as the number of element in the domain
- Denote  $m \coloneqq |f(n)|$  as the number of element in the domain



- If |n| > |f(n)|, then the function is a non-injective surjective function (not a bijection)
- If |n| = |f(n)|, then the function is a injective surjective function (bijection)
- If |n| < |f(n)|, then the function is an injective non-surjective function (not a bijection).

# Pigeonhole principle example .. (1/2)

- E.g. Put five pigeons into two pigeonholes A, B, then one of the pigeonhole will has at least 3 pigeons.
  - By enumeration:

 $\begin{array}{ll} AAAAA & A \mbox{ has } 5 > 3 \mbox{ pigeons} \\ AAAAB & A \mbox{ has } 4 > 3 \mbox{ pigeons} \\ AAABB & A \mbox{ has } 3 = 3 \mbox{ pigeons} \\ AABBB & B \mbox{ has } 3 = 3 \mbox{ pigeons} \\ ABBBB & B \mbox{ has } 4 > 3 \mbox{ pigeons} \\ BBBBB & B \mbox{ has } 5 > 3 \mbox{ pigeons} \end{array}$ 

- E.g. In a group of 13 people, we have 2 or more who are born in the same month.
- Number of pigeons: 13 (number of people)
- Number of pigeonholes: 12 (number of months)
- E.g. In a group of n > 1 people, each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.
  - Number of pigeons: *n* (number of people)
- Number of pigeonholes: n-1 (range of number of handshakes)
- E.g. For any choice of 10 numbers in  $\{1, 2, \dots, 19\}$ , there are two that add up to 19.
- we partition  $\{1, 2, \ldots, 19\}$  into 9 subsets:  $\{1, 18\}, \{2, 17\}, \{3, 16\}, \ldots, \{9, 10\}.$
- we are picking 10 numbers, by the pigeonhole principle, there must be two chosen numbers lie in the same bracket

# Pigeonhole principle example .. (2/2)

- E.g. A class of 20 students got their exam scores, I told them the class average for the test was 8 out of a max of 10. Then someone in the class must have scored at least an 8/10.
  - **Proof by contradiction** Let  $x_i$  be the scores of the student.
  - For contradiction, assume the statement is false. I.e., nobody get more than 8/10. This means  $x_i < 8$  for all i.
  - The average of all 20 students is then

class average 
$$= \frac{x_1 + x_2 + \dots + x_{20}}{20} \stackrel{\text{assumption}}{<} \frac{8 + 8 + \dots + 8}{20} = \frac{160}{20} = 8.$$

This contradicts with  $\_$ , therefore assumption is false, and someone get more than 8/10.

- E.g. For any 5 points in the unit square  $[0,1] \times [0,1]$ , there are two points that are within  $\frac{1}{\sqrt{2}}$  distance of each other
  - We partition  $[0,1] \times [0,1]$  into 4 quadrants.
  - By the pigeonhole principle, two of the points are in the same quadrant.

• The farthest two points can be in the same quadrant is the length of the diagonal of the square, which is  $\frac{1}{\sqrt{2}}$ . Now you see pigeonhole principle is *existential*: it only gives the information of "there exists in an object that has property X", it does not specify which object has the property X.

# The relationship of 6 people

# Not in exam

- **Theorem** Suppose there are 6 people  $\{1, 2, 3, 4, 5, 6\}$ , and for each pair (i, j) there is a relationship of  $\{\texttt{friends}, \texttt{stranger}\}$ . Then there will be at least 3 people that they are friends of each other or they are strangers.
- The above two situations **always** hold no matter how you pick 6 people in this world. This statement does not hold if you consider less than 6 people.
- Ramsey number R(3,3) = 6
- Proved by pigeonhole principle on a complete graph: 5 edges, 2 group, 3 connections
- Claim: for A, at least 3 edges on friends / stranger
- Pigeon: 3
- Pigeonhole:  $\lfloor \frac{5}{2} \rfloor$


# What's the big deal of pigeonhole principle?

#### Not in exam

- Pigeonhole principle, founded by Peter Gustav Lejeune Dirichlet, gives us a guarantee on what can happen in the "worst case scenario".
- Pigeonhole principle is a special case of Ramsey's theory.
- Ramsey theory: "how big must some structure be to guarantee a particular property holds?"
- "Complete disorder is impossible" Theodore S. Motzkin
- Frank Ramsey, who proposed the Ramsey's theory in 1928 (at age 25), is a friend of Ludwig Wittgenstein (who invented the *truth table*). Wittgenstein is PhD a student of Bertrand Russell (logic).



Figure: Ramsey, Wittgenstein and Russell

• Ramsey theory is an entire branch of mathematics that is too difficult (at least for me)

#### Pigeonhole principle in computer science

Not in exam

- Pigeonhole principle is not a theorem but rather a proof-technique.
- You will not see the power of pigeonhole principle in this course. You will only experience its power in later course.
- E.g. All list with more than  $n^2$  distinct numbers has a monotone sublist of length greater than n.
- e.g. n = 3, so we need to build a list of at least  $n^2 = 9$  distinct numbers.
- Say we randomly generate  $10 > n^2 = 9$  distinct integers as

 $\texttt{list} = \{1, 10, 2, 9, 3, 8, 4, 7, 5, 6\}$ 

A sublist of list, in which you can skip elements, is

 $sublist = \{1, 2, 3, 4, 5, 6\}, |sublist| = 6 > 3 =: n.$ 

- E.g. Any comparison-based sorting algorithm must perform  $\Omega(n \log n)$  comparisons to sort n elements in the worst case.
- Proof by contradiction: if the algorithm makes fewer comparisons than this, by pigeonhole principle, there
  must be some pair of inputs that the algorithm wouldn't be able to distinguish, since there are more possible
  inputs than configurations of the algorithm.
- E.g. Any hash function with more inputs than outputs will necessarily have collision.

# Summary

- 1. Inclusion-exclusion principle  $|A \cup B| = |A| + |B| |A \cap B|$
- 2. Counting by enumeration: list all possible outcome, draw a tree to help visualize
- 3. Sum rule as special case of inclusion-exclusion principle  $|\mathcal{P}\cup\mathcal{Q}|=|\mathcal{P}|+|\mathcal{Q}|$
- 4. Product rule as application of Cartesian product  $|\mathcal{P} \times \mathcal{Q}| = |\mathcal{P}| \cdot |\mathcal{Q}|$
- 5. Subtraction rule as application of complement:  $A \subset S$  then for  $A^c \setminus S$ , we have  $|A^c| = |S| |A|$
- 6. Floor function and counting number of divisible integers
- 7. A permutation is an ordered arrangement of a set of objects
- 8. The number of permutations of n distinct objects is  $n! := n(n-1)(n-2)\cdots 1$  factorial
- 9. Generalized permutation of n objects, possible non-distinct is  $rac{n!}{n_1!\ldots n_r!}=inom{n}{n_1,n_2,...,n_r}$

division rule

- 10. A combination is an unordered arrangement of a set of objects
- 11. Binomial coefficient *n*-choose-*k* without repetition is  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$
- 12. *n*-choose-*k* with repetition is  $\binom{n+k-1}{k}$
- 13. Counting using binomial and multinomial coefficients
- 14. Counting via bijection
- 15. Counting via generating function
- 16. Solving recursion by generating function
- 17. Pigeonhole principle

### Contents

ounting by set theory Inclusion-exclusion principle Enumeration, sum rule & product rule

Counting by binomial coefficient Permutation, factorial and division rule Combination and binomial Binomial expansion Trinomial & multinomial

Advanced counting techniques

Counting via bijection Counting via generating function / z-transform Recursion, partial fraction and generating function Pigeonhole principle

Advanced<sup>2</sup> counting techniques Gosper's algorithm Hypergeometric function Gosper's algorithm for crazy sums

• 
$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{k+m}{2n} = \binom{2m}{2n}$$

• 
$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} = (-4)^n$$

• 
$$\sum_{k=0}^{n} \frac{k^5 - 52}{k!} = -\frac{n^4 + 5n^3 + 14n^2 + 31n + 52}{n!}$$

- Gosper's algorithm: Bill Gosper (the founder of hacker) discovered a way to solve these crazy sum.
- Techniques
- partial fraction decomposition
- telescoping series
- Hypergeometric function

Gosper's algorithm for crazy sums

• 
$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{k+m}{2n} = \binom{2m}{2n}$$

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$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} = (-4)^n$$

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- Gosper's algorithm: Bill Gosper (the founder of hacker) discovered a way to solve these crazy sum.
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- partial fraction decomposition
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#### Pochhammer symbol

- Pochhammer falling factorial  $x^{\underline{n}} = x(x-1)(x-2)...(x-n+1) = \frac{x!}{(x-n)!}$
- we define  $x^{\underline{0}} \coloneqq 1$
- Pochhammer raising factorial  $x^{\overline{n}} = x(x+1)(x+2)...(x+n-1)$
- we define  $x^{\overline{0}} \coloneqq 1$
- Note:  $1^{\overline{n}} = n!$
- Hypergeometric function For  $a_i$  integer and  $b_i$  natural number

$$_{n}F_{m}(a_{1}, a_{2}, ..., a_{n}; b_{1}, b_{2}, ..., b_{m}; z) = \sum_{k=0}^{\infty} \frac{a_{1}^{k} ... a_{n}^{k}}{b_{1}^{k} ... b_{m}^{k}} \frac{z^{k}}{k!}$$

- n means how many parameters on the top
- $\bullet \ m$  means how many parameters at the bottom
- the top parameters are  $a_1, a_2, ...$
- the bottom parameters are  $b_1, b_2, ...$

All functions are just special cases of hypergeometric function Not in exam • E.g. n = m = 1 and  $a_1 = b_1 = 1$ 

$${}_{1}F_{1}(a_{1};b_{1};z) = {}_{1}F_{1}(1;1;z) = \sum_{k=0}^{\infty} \frac{1^{\overline{k}}}{1^{\overline{k}}} \frac{z^{k}}{k!}$$
  
$$= \sum_{k=0}^{\infty} \frac{z^{k}}{k!}$$
$$= e^{z}$$

• E.g. 
$$n = 2, m = 1$$
 and  $a_1 = a_2 = b_1 = 1$   
 ${}_2F_1(a_1, a_2; b_1; z) = {}_2F_1(1, 1; 1; z) = \sum_{k=0}^{\infty} \frac{1^{\overline{k}_1 \overline{k}}}{1^{\overline{k}}} \frac{z^k}{k!}$   
 $= \sum_{k=0}^{\infty} 1^{\overline{k}} \frac{z^k}{k!}$   
 $= \sum_{k=0}^{\infty} k! \frac{z^k}{k!}$   
 $= \sum_{k=0}^{\infty} z^k$   
 $= \frac{1}{1-z}$ 

$$E \text{ E.g. } n = 2, m = 1, a_1 = a, a_2 = 1, b_1 = 1$$

$${}_2F_1(a, 1; 1; z) = \sum_{k=0}^{\infty} \frac{a^{\overline{k}} 1^{\overline{k}}}{1^{\overline{k}}} \frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty} a^{\overline{k}} \frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty} a(a+1) \cdots (a+k-1) \frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty} {\binom{n+k-1}{k}} z^k$$

$$= \frac{1}{(1-z)^a}$$

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Why hypergeometric function: an ultimate tool

• How do one solve 
$$\sum_{k=0}^{n} {\binom{n}{k}}^{2} {\binom{2k}{k}}$$
?  
•  $1^{\overline{k}} = 1 \cdot 2 \dots k = k!$   
• Double factorial  
•  $1^{\overline{k}} = 1 \cdot 2 \dots k = k!$   
• Double factorial  
•  $1^{\overline{k}} = 1 \cdot 2 \dots k = k!$   
• Double factorial  
•  $1^{\overline{k}} = 1 \cdot 2 \dots k = k!$   
• Double factorial  
•  $2k := 0$   $\binom{n}{k}^{2} {\binom{2k}{k}}$  =  $\sum_{k=0}^{n} \left(\frac{n!}{k!(n-k)!}\right)^{2} \cdot \frac{(2k)!}{(k!)^{2}} + \frac{1}{(k!)!}$   
=  $\sum_{k=0}^{n} \frac{(n!)^{2}(2k)!}{(k!)^{n}(n-k)!^{2}} \cdot \frac{1}{k!k!k!}$   
=  $\sum_{k=0}^{n} \frac{(n!)^{2}}{(n-k)!^{2}} \cdot \frac{(2k)!}{k!} \cdot \frac{1}{1\overline{k}1\overline{k}k!}$   
=  $\sum_{k=0}^{n} \frac{(n!)^{2}}{(n-k)!^{2}} \cdot \frac{(2k)!!}{k!} \cdot \frac{1}{1\overline{k}1\overline{k}k!}$   
=  $\sum_{k=0}^{n} \frac{(-n)^{\overline{k}}(-n)^{\overline{k}}}{1\overline{k}1\overline{k}} \cdot (\frac{1}{2})^{\overline{k}} \cdot \frac{4^{k}}{k!}$   
=  $\sum_{k=0}^{n} \frac{(-n)^{\overline{k}}(-n)^{\overline{k}}}{1\overline{k}1\overline{k}} \cdot (\frac{1}{2})^{\overline{k}}} \cdot \frac{4^{k}}{k!}$   
=  $(n(n-1)\dots(n-k+1)) (n(n-1)\dots(n-k+1))$   
=  $(-1)^{k}(-n)(-n+1)\dots(-n+k-1))$   
 $\cdot ((-1)^{k}(-n)(-n+1)\dots(-n+k-1))$   
 $\cdot ((-1)^{k}(-n)(-n+1)\dots(-n+k-1))$   
 $\cdot ((-1)^{k}(-n)(-n+1)\dots(-n+k-1))$   
 $\cdot ((-1)^{k}(-n)(-n+1)\dots(-n+k-1))$ 

### Things we didn't cover

- Stirling, Bell, Eulerian, Catalan number
- Negative binomial, Gamma function and generalized binomial
- Chain, Antichain, Sperner's Theorem
- Systems of distinct representatives
- Latin Squares
- Symmetric group
- Burnside's Theorem
- Advanced / research level
- Analytic Combinatorics
- Additive combinatorics
- Extremal combinatorics, Ramsey theory
- Arithmetic combinatorics