# Key concepts for COMP1215 and reading list

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# 1 Key concepts in combinatorics

## 1.1 Basic counting techniques from set theory

1. Given a set $A$, the number of elements of $A$ is denoted as $|A|$
2. Inclusion-exclusion principle $|A \cup B| = |A| + |B| - |A \cap B|$
3. Counting by enumeration: list all possible outcome. Draw tree to visualize.
4. Sum rule as a special case of inclusion-exclusion principle: $|A \cup B| = |A| + |B|$
5. Product rule as application of Cartesian product $|A \times B| = |A| \cdot |B|$
6. Subtraction rule as application of complement and disjoint set: If $A \subset S$ then for $A^c = S \setminus A$, we have $|A^c| = |S| - |A|$
7. Floor function $\lfloor x \rfloor \coloneqq \max\{m \in \mathbb{Z} \mid m \leq x\}$, and counting number of divisible integers

## 1.2 Basic counting techniques from binomial coefficient

1. Factorial $k! = \begin{cases} 1 & k = 0 \\ k(k-1)! & k > 0 \end{cases}$
2. binomial coefficient $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$
3. trinomial coefficient $\binom{n}{i,j,k} = \dfrac{n!}{i!j!k!}$ and multinomial coefficient $\binom{n}{k_1,\cdots,k_r} = \dfrac{n!}{k_1!\cdots k_r!}$
4. Definition of permutation: an ordered arrangement of a set of distinct objects
5. Definition of combination: an unordered arrangement of a set of distinct objects
6. $n!$ = number of permutation of $n$-set of distinct object
7. $\binom{n}{k}$ = number of $n$-choose-$k$ combination of a $n$-set of distinct objects
8. $\binom{n}{k_1,k_2,...,k_r}$ = number of permutation of a $n$-set with $(k_1,k_2,...,k_r)$ repeated object
9. $\binom{n+k-1}{k}$ = number of $n$-choose-$k$ combination of a $n$-set with $k$ repeated objects

| | order important | order not important |
|---|---|---|
| no repetition | permutation $n!$ | combination $\binom{n}{k}$ |
| repetition | generalized permutation $\binom{n}{n_1,n_2\dots,n_r}$ | generalized combination $\binom{n+k-1}{k}$ |

10. Number of shortest lattice paths from $\begin{bmatrix}0\\0\end{bmatrix}$ to $\begin{bmatrix}m\\n\end{bmatrix}$ is $\binom{m+n}{m}$
11. Binomial expansion $(x+y)^n$
12. Solving trinomial expansion using binomial expansion
13. Properties of binomial coefficient, Pascal's triangle and Fibonacci sequence

### 1.3 Advanced counting techniques

- Counting by bijection
  - Sometimes counting directly on $X$ is difficult
  - Construct a function $x \mapsto f(x)$ that $f : X \to Y$ is bijective
  - Count $Y$ instead
- Counting by polynomial coefficient $[x^n]$ on the outcome of tossing dice
- Geometric series: $1 + x + x^2 + ... + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ for $x \neq 1$ and $1 + x + x^2 + \cdots = \dfrac{1}{1 - x}$ for $|x| < 1$
- $\left(1 + x + x^2 + ...\right)^n = \displaystyle\sum_{r=0}^{\infty} \binom{r + n - 1}{r} x^r$
- Generating function technique $G(x)$
- Partial fraction decomposition of $\dfrac{f(x)}{g(x)}$
- Solving recursion by generating function
- Pigeonhole principle

## 2 Key concepts in probability

### 2.1 Basic probability from set theory

1. The notation of sample space, event and probability
2. The three probability axioms
3. Complementary event, mutually exclusive events

### 2.2 Random variable

1. Joint probability, marginal probability, conditional probability
2. Independent random variables
3. Expected value $\mathbb{E}[X]$ and $\mathbb{E}[f(X)]$
4. Expectation is linear $\mathbb{E}\Big[aX + bY + c\Big] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$
5. Variance $\mathbb{V}[X] = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)^2\Big] = \mathbb{E}[X^2] - \Big(\mathbb{E}[X]\Big)^2$
6. Variance quadratic formula $\mathbb{V}\Big[aX \pm bY + c\Big] = a^2\mathbb{V}[X] \pm 2ab\,\mathrm{cov}(X, Y) + b^2\mathbb{V}[Y]$
7. $\mathrm{cov}(X, Y) = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\Big]$
   how to remember: replace one $X$ in $\mathbb{V}[X]$ by $Y$
8. $\mathrm{corr}(X, Y) = \dfrac{\mathrm{cov}(X, Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}}$
9. The three examples $\mathbb{E}[X + Y], \mathbb{E}[XY], \mathbb{E}[(X, Y)]$

## 3 Key concepts in statistics

### 3.1 Probability distribution function

1. Probability = area under the curve of PDF
2. Crazy things about probability for continuous random variable
   (a) $\mathbb{P}(X \leq a) = \mathbb{P}(X < a)$ because $\mathbb{P}(X = a) = 0$ this is not true for discrete random variable
   (b) Zero probability $\neq$ impossible to occur

3. Bernoulli distribution

   - $\mathcal{X} = \{0, 1\}$
   - $X \sim \mathrm{Ber}(\theta)$ then $\mathbb{P}(X = x|\theta) = p(x|\theta) = \theta^x(1-\theta)^{1-x}$
   - It is used for modelling binary event

4. Binomial distribution

   - $\mathcal{X} = \{0, 1, ..., n\}$
   - $X \sim \mathrm{Bin}(n, \theta)$ then $\mathbb{P}(X = x|n, \theta) = p(x|n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$
   - It is used for modelling number of success in $n$ binary event

5. Poisson distribution

   - $\mathcal{X} = \{0, 1, 2, ...\}$
   - $X \sim \mathrm{Poi}(\lambda)$ then $\mathbb{P}(X = x|\lambda) = p(x|\lambda) = \dfrac{\lambda^x e^-\lambda}{x!}$
   - It is used for modelling time interval event

## 3.2 Normal random variable

1. Normal random variable

   - $\mathcal{X} = \mathbb{R}$
   - $X \sim \mathcal{N}(\mu, \sigma^2)$ then $\mathbb{P}(X = x|\mu, \sigma^2) = p(x|\mu, \sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
   - $\mathbb{E}[X] = \mu$
   - $\mathbb{V}[X] = \sigma^2$

2. Standard Normal random variable $Z \sim \mathcal{N}(0, 1)$ and $\mathbb{P}(Z = z|0, 1) = p(z|0, 1) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$

3. If $Z \sim \mathcal{N}(0, 1)$ then $\mathbb{P}(a \leq Z \leq b) = \displaystyle\int_a^b \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz = \frac{\mathrm{erf}(b) - \mathrm{erf}(a)}{2}$

4. Calculating error function $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$ by computer

5. Standardization $X \sim \mathcal{N}(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

## 3.3 Point estimation

- Knowing the difference between population parameter and estimator
- sample mean $\overline{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$
- If $X_1, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$ then sample mean $\overline{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$
- sample variance $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2$ is a biased estimator
- unbiased estimator of variance $s^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2$

### 3.4 Confidence interval

1. The $100(1-\alpha)\%$ confidence interval of population mean, known variance is

$$T_\alpha = \left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

   where $Z \sim \mathcal{N}(0,1)$. For 95% confidence interval, $\alpha = 0.05$ and $z_{\alpha/2} = 1.96$ (computed by solving error function)

2. The $100(1-\alpha)\%$ confidence interval of population mean, unknown variance is

$$T_\alpha = \left[\overline{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}},\ \overline{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right]$$

   where $T \sim T(n-1)$. The value $t_{\alpha/2,n-1}$ has to be solved by computer.

3. The $100(1-\alpha)\%$ confidence interval of difference of two population mean, known variance is

$$T_\alpha = \left[\overline{x}_A - \overline{x}_B - z_{\alpha/2}\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}},\ \overline{x}_A - \overline{x}_B + z_{\alpha/2}\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}\right]$$

   we use the same formula for the case of unknown variance, by replacing $\sigma^2$ with $s^2$

### 3.5 Hypothesis testing

1. We take the null hypothesis (innocent) as the default position, and we use data to get evidence aginst the null
2. $p$-value is the probability $\mathbb{P}(\text{data}|H_0)$
   - small probability means we have lots of evidence against the null, so null is probably false
   - large probability means we do not have enough evidence against the null, no conclusion
   - large probability does not mean $H_0$ is true
3. For $\begin{array}{ccc} H_0 & : & \mu = \mu_0 \\ & \text{vs} & \\ H_A & : & \mu \neq \mu_0 \end{array}$, the $p$-value is

$$p = 2\mathbb{P}\Big(Z < -|z^*|\Big) = 2\int_{-\infty}^{z^*} p(z)dz \quad \text{where} \quad z^* = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

## 4 Key concepts in basic graph theory

1. What is graph, vertex, edge
2. Directed vs undirected, weighted, simple vs multiedge
3. Representation of a graph: VVEEMAD
4. Degree and Handshaking lemma
5. All graph has even number of odd vertices
6. Walk, trail, path, cycle
7. Number of walks in a graph

## 5 So, you want a reading list?

- **Discrete Mathematics and Its Applications, 7th Edition** Author: Kenneth Rosen
  QA39 ROS in Hartley Library
  - Chapter 6 Counting page 385-444
  - Chapter 7 Discrete Probability page 445-494
    Skip Section 7.3
  - Chapter 8 Advanced Counting Techniques
    Section 8.4 Generating Functions page 537-552
    Section 8.5 Inclusion-Exclusion page 552-558

- Chapter 10 Graphs page 641-678

Older or newer version are also ok

- **Introductory Statistics, 3rd edition** Author: Sheldon Ross
  The library has the ebook (click "view ebook" on the right)
  Basically you can read everything from page 1 to page 502
  - Chapter 1-3 are "high school level" you can skip or read
  - Chapter 4 Probability you can skip or read
  - Chapter 5 Discrete Random Variables page 209-page 260
    Focus on 5.1-5.5
  - Chapter 7 Distributions of Sampling Statistics you can skip or read
    You can read the whole chapter, skip the whole chapter, or just focus on 7.3 and 7.4
  - Chapter 6 Normal Random Variables page 261-296
  - Chapter 8.5-8.7 page 347-386
  - Chapter 9 Testing Statistical Hypotheses page 387-442
    Focus on 9.1-9.4
  - Chapter 10 Hypothesis Tests Concerning Two Populations page 443-502
    Focus on 10.1-10.3
- **Foundations of Computer Science, 2nd Revised Edition** Compilded by Powel Sobocinski
  QA39 SOB in Hartley library, 36 copies
  - Chapter 6.1-6.3 page 505-539
  - Chapter 6.5-6.8 page 545-569
  - Chapter 8.1-8.2 page 617-640
  - Chapter 8.5-8.6 page 653-665
  - Chapter 9.1-9.2 page 681-694
- **Schaum's outlines Discrete Mathematics, 3rd edition** Author: Seymour Lipschutz
  QA43 LIP in Hartley library, have ebook
  - Chapter 5 to Chapter 9 page 88-204

# 6 Advanced / challenging books

- **Concrete Mathematics A foundation for Computer Science, 2nd edition** author: Graham, Knuth, Patashnik
  QA39 GRA in Hartley library
  - Chapter 2 Sums page 21-25, page 30-41
  - Chapter 3.1 Floors and Ceilings page 67-78
  - Chapter 5 Binomial Coefficients
    * Ch5.1 page 153-172
    * Ch5.4 page 196-204
  - Chapter 7 Generating Functions page 320-380
  - Chapter 8 Discrete Probability page 381-38
- **Counting : the art of enumerative combinatorics** author: George E Martin
  QA164.8 MAR

# 7 Very challenging books

- **102 Combinatorial Problems: From the Training of the USA IMO Team**
- **Enumerative combinatorics. Volume 1** Author: Richard P Stanley
  ebook avaliable in the library

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