Madbook for COMP1215 (Combinatorics, Probability, Statistics and Graph)

Author: Andersen Ang (ECS, University of Southampton, UK) Homepage: angms.science

First draft: June 1, 2023 version: January 15, 2024

This document is the exercise book for the course COMP1215 Foundations of Computer Science (Semester 1 of the academic year 2023-2024) on the part of Combinatorics, Graph, Probability and Statistic

Contents

1 Mathematical induction

1.1 Problems

1. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} i^1 = \frac{1}{2}\Big(n^2 + \frac{2}{2}n\Big)$ $\qquad 1+2+\cdots+n = \frac{1}{2}\Big(n^2+\frac{2}{2}n\Big)$

2. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} i^2 = \frac{1}{3}\Big(n^3 + \frac{3}{2}n^2 + \frac{3}{6}n\Big)$ $\qquad 1^2+2^2+\cdots+n^2 = \frac{1}{3}\Big(n^3+\frac{3}{2}n^2+\frac{3}{6}n\Big)$

3. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} i^3 = \frac{1}{4}\Big(n^4 + \frac{4}{2}n^3 + \frac{6}{6}n^2\Big)$ $\qquad 1^3+2^3+\cdots+n^3 = \frac{1}{4}\Big(n^4+\frac{4}{2}n^3+\frac{6}{6}n^2\Big)$

4. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} i^4 = \frac{1}{5}\Big(n^5 + \frac{5}{2}n^4 + \frac{10}{6}n^3 + 0n^2 - \frac{5}{30}n\Big)$ $\qquad 1^4+2^4+\cdots+n^4 = \frac{1}{5}\Big(n^5+\frac{5}{2}n^4+\frac{10}{6}n^3-\frac{5}{30}n\Big)$
   (If you are interested to know more, google "Faulhaber's formula")

5. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$ $\qquad 1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)!-1$

6. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$ $\qquad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

7. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$ $\qquad \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

8. Let $n \in \mathbb{N}$, prove $\sum_{i=1}^{n} \frac{i+2}{i(i+1)(2^i)} = 1 - \frac{1}{(n+1)2^n}.$ $\qquad \frac{3}{1\cdot 2\cdot 2}+\frac{4}{2\cdot 3\cdot 2^2}+\cdots+\frac{n+2}{n(n+1)2^n} = 1-\frac{1}{(n+1)2^n}$

9. Let $n \in \mathbb{N}$, prove $x^n - 1$ is divisible by $x-1$.
   Hint: a polynomial $p(x)$ is divisible by a polynomial $q(x)$ if there is a polynomial $r(x)$ such that $p(x)$ can be written as $q(x)r(x)$.

10. Let $n \in \mathbb{N}$, prove $n^3 + 2n$ is divisible by 3.

11. Let $n \in \mathbb{N}$, prove $17n^3 + 103n$ is divisible by 6.

12. Let $n \in \mathbb{N}$, prove $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

13. Let $n \in \mathbb{N}$ that is odd, prove $2^n + 1$ is divisible by 3.

14. Let $n \in \mathbb{N}$ that is odd, prove $n^2 - 1$ is divisible by 8.

15. Let $n \in \mathbb{N}$ that is odd, prove $n^4 - 1$ is divisible by 16.

16. Let $n \in \mathbb{N}$ and $r \neq 1$, prove $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

17. Let $n \in \mathbb{N}$, prove $S(n) = 1+3+5+\cdots+(2n-1) = n^2$. That is, the sum of the first n odd numbers is n^2.

18. Let $n \in \mathbb{N}$, prove the sum $S(n) = 8+13+18+23+\cdots+(3+5n)$ has a closed-form formula $2.5n^2+5.5n$.

19. Let $n \in \mathbb{N}$ and $1+x>0$, prove $(1+x)^n \geq 1+nx$.

20. Let $n \in \mathbb{N}$ and $n \geq 2$, prove $n^2 + 4n < 4^n$.

21. Let $n \in \mathbb{N}$, prove $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} < 1.$

22. Let $n \in \mathbb{N}$, prove $\sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{n}}}} < 3.$

23. Let $n \in \mathbb{N}$ that is larger than 4 (4 included), prove $n! > 2^n$.

24. Let $n \in \mathbb{N}$ that is larger than 2 (2 included), prove $\prod_{i=2}^{n}\Big(1-\frac{1}{i^2}\Big) = \frac{n+1}{2n}$

25. Let $n \in \mathbb{N}$ that is larger than 2 (2 included), prove $n^{1/n} < 2 - \dfrac{1}{n}$

26. Let $n \in \mathbb{N}$. Prove the binomial theorem using induction.

1.2 Solution

1. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^1 = \frac{1}{2}\Big(n^2 + \frac{2}{2}n\Big)$

   (a) Base case: $n = 1$.
   LHS (Left hand side) $= 1$
   RHS (right hand side) $\dfrac{1}{2}(1^2 + 1) = 1$
   LHS $=$ RHS so base case is true.

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i = \frac{1}{2}\Big(k^2 + \frac{2}{2}k\Big) \tag{H}$$

   (c) Case $n = k + 1$

   $$\begin{aligned} LHS = \sum_{i=1}^{k+1} i \quad &= \quad \Big(\sum_{i=1}^{k} i\Big) + k + 1 \\ &\overset{(H)}{=} \quad \frac{1}{2}\Big(k^2 + \frac{2}{2}k\Big) + k + 1 \\ &= \quad \frac{1}{2}\Big(k^2 + \frac{2}{2}k + 2k + 2\Big) \\ &= \quad \frac{1}{2}\Big(k^2 + 2k + 1 + \frac{2}{2}k + 1\Big) \\ &= \quad \frac{1}{2}\Big((k+1)^2 + \frac{2}{2}(k+1)\Big) = RHS \end{aligned}$$

   So the case for $n = k + 1$ is true.

2. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{3}\Big(n^3 + \frac{3}{2}n^2 + \frac{3}{6}n\Big)$

   (a) Base case $n = 1$: LHS $= 1^2 = 1 = \dfrac{1}{3}\cdot 3 = \dfrac{1}{3}\Big(1 + \dfrac{3}{2} + \dfrac{3}{6}\Big) = \dfrac{1}{3}\Big(1^3 + \dfrac{3}{2}1^2 + \dfrac{3}{6}1\Big) =$ RHS

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i^2 = \frac{1}{3}\Big(k^3 + \frac{3}{2}k^2 + \frac{3}{6}k\Big) \tag{H}$$

   (c) Case $n = k + 1$

   $$\begin{aligned} LHS \quad &= \quad \textstyle\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \overset{(H)}{=} \dfrac{1}{3}\Big(k^3 + \dfrac{3}{2}k^2 + \dfrac{3}{6}k\Big) + k^2 + 2k + 1 = \dfrac{1}{3}\Big(k^3 + \dfrac{3}{2}k^2 + \dfrac{3}{6}k + 3k^2 + 6k + 3\Big) \\ RHS \quad &= \quad \dfrac{1}{3}\Big((k+1)^3 + \dfrac{3}{2}(k+1)^2 + \dfrac{3}{6}(k+1)\Big) \end{aligned}$$

   We show LHS $=$ RHS by showing LHS - RHS $= 0$.

   $$\begin{aligned} 3LHS - 3RHS \quad &= \quad \Big(k^3 + \tfrac{3}{2}k^2 + \tfrac{3}{6}k + 3k^2 + 6k + 3\Big) - \Big((k+1)^3 + \tfrac{3}{2}(k+1)^2 + \tfrac{3}{6}(k+1)\Big) \\ &= \quad \Big(k^3 + \tfrac{8}{2}k^2 + \tfrac{39}{6}k + 3\Big) - \Big((k^3 + 3k^2 + 3k + 1) + \tfrac{3}{2}(k^2 + 2k + 1) + \tfrac{3}{6}(k+1)\Big) \\ &= \quad 0 \end{aligned}$$

   So the case for $n = k + 1$ is true.

3. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^3 = \frac{1}{4}\Big(n^4 + \frac{4}{2}n^3 + \frac{6}{6}n^2\Big)$

   (a) Base case $n = 1$: LHS$= 1^3 = 1 = \dfrac{1}{4}\cdot 4 = \dfrac{1}{4}\Big(1^4 + \tfrac{4}{2}1^3 + \tfrac{6}{6}1^2\Big)$ =RHS

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i^3 = \frac{1}{4}\Big(k^4 + \frac{4}{2}k^3 + \frac{6}{6}k^2\Big) \tag{H}$$

(c) Case $n = k+1$

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \overset{(H)}{=} \frac{1}{4}\Big(k^4 + \frac{4}{2}k^3 + \frac{6}{6}k^2\Big) + (k+1)^3 \\ RHS &= \frac{1}{4}\Big((k+1)^4 + \frac{4}{2}(k+1)^3 + \frac{6}{6}(k+1)^2\Big) \end{aligned}$$

We show LHS = RHS by showing LHS - RHS $= 0$.
Work out the details of $4LHS - 4RHS = 0$ (do it yourself) shows the case for $n = k+1$ is true.

4. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^4 = \frac{1}{5}\Big(n^5 + \frac{5}{2}n^4 + \frac{10}{6}n^3 + 0n^2 - \frac{5}{30}n\Big)$

   (a) Base case $n = 1$: LHS= $1^4 = 1 = \dfrac{1}{5}\cdot 5 = \dfrac{1}{5}\Big(1^5 + \frac{5}{2}1^4 + \frac{10}{6}1^3 + 0\cdot 1^2 - \frac{5}{30}\cdot 1\Big)$ =RHS

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

$$\sum_{i=1}^{k} i^4 = \frac{1}{5}\Big(k^5 + \frac{5}{2}k^4 + \frac{10}{6}k^3 + 0k^2 - \frac{5}{30}k\Big) \tag{H}$$

   (c) Case $n = k+1$

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^4 = \sum_{i=1}^{k} i^4 + (k+1)^4 \overset{(H)}{=} \frac{1}{5}\Big(k^5 + \frac{5}{2}k^4 + \frac{10}{6}k^3 + 0k^2 - \frac{5}{30}k\Big) + (k+1)^4 \\ RHS &= \frac{1}{5}\Big((k+1)^5 + \frac{5}{2}(k+1)^4 + \frac{10}{6}(k+1)^3 + 0(k+1)^2 - \frac{5}{30}(k+1)\Big) \end{aligned}$$

   We show LHS = RHS by showing LHS - RHS $= 0$.
   Work out the details of $5LHS - 5RHS = 0$ (do it yourself) shows the case for $n = k+1$ is true.

5. Base case $1 \cdot 1! = 2! - 1$.

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$.

   Case $n = k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} i \cdot i! &= \sum_{i=1}^{k} i \cdot i! + (k+1)\cdot(k+1)! \\ &\overset{H}{=} (k+1)! - 1 + (k+1)\cdot(k+1)! \\ &= \Big(1 + (k+1)\Big)(k+1)! - 1 \\ &= (k+2)! - 1 \end{aligned}$$

6. Base case: for $n = 1$ we have $\dfrac{1}{1(2)} = \dfrac{1}{2} = \dfrac{1}{1+1}$

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$.

   At case $n = k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\ &\overset{(H)}{=} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \end{aligned}$$

7. Base case: for $n = 1$ we have $\dfrac{1}{1(3)} = \dfrac{1}{3} = \dfrac{1}{2(1)+1}$

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$.

At case $n = k+1$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&\overset{(H)}{=} \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{1}{2k+1}\Big(k + \frac{1}{2k+3}\Big) = \frac{1}{2k+1}\frac{2k^2+3k+1}{2k+3} = \frac{1}{2k+1}\frac{(2k+1)(k+1)}{2k+3} = \frac{k+1}{2(k+1)+1}
\end{aligned}$$

8. Base case: for $n = 1$, LHS$= \dfrac{3}{1\cdot 2\cdot 2} = 1 - \dfrac{1}{2\cdot 2^1}$ =RHS
   Induction hypothesis:
   $$\sum_{i=1}^{k} \frac{i+2}{i(i+1)(2^i)} = 1 - \frac{1}{(k+1)2^k}. \tag{H}$$
   Case $n = k+1$
   $$\begin{aligned}
   \sum_{i=1}^{k+1} \frac{i+2}{i(i+1)(2^i)} &= \sum_{i=1}^{k} \frac{i+2}{i(i+1)(2^i)} + \frac{k+1+2}{(k+1)(k+1+1)(2^{k+1})} \\
   &\overset{H}{=} 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)(2^{k+1})} \\
   &= 1 + \frac{1}{(k+1)2^k}\Big(-1 + \frac{k+3}{(k+2)2}\Big) \\
   &= 1 + \frac{1}{(k+1)2^k}\Big(\frac{-2(k+2)}{2(k+2)} + \frac{k+3}{2(k+2)}\Big) \\
   &= 1 + \frac{1}{(k+1)2^k}\frac{-2(k+2)+k+3}{2(k+2)} \\
   &= 1 + \frac{1}{(k+1)2^{k+1}}\frac{-2k-4+k+3}{k+2} \\
   &= 1 + \frac{1}{(k+1)2^{k+1}}\frac{-k-1}{k+2} \\
   &= 1 - \frac{1}{(k+2)2^{k+1}}
   \end{aligned}$$

9. Base case: for $n = 1$ we have $x-1$ divisible by $x-1$.
   Induction hypothesis: assume $x^k - 1$ is divisible by $x-1$. That is, we have
   $$x^k - 1 = (x-1)q(x) \tag{H}$$
   for some polynomial $q(x)$.
   At case $n = k+1$, we have
   $$\begin{aligned}
   x^{k+1} - 1 &= x^{k+1} - (x-1) - 1 + (x-1) \\
   &= x^{k+1} - x + 1 - 1 + x - 1 \\
   &= x(x^k - 1) + x - 1 \\
   &\overset{H}{=} x(x-1)q(x) + x - 1 \\
   &= (xq(x)+1)(x-1) \\
   &= q'(x)(x-1)
   \end{aligned}$$
   Therefore, $x^{k+1} - 1$ is divisible by $x-1$.

10. Base case: for $n = 0$ we have 0 divisible by 3.
    Induction hypothesis: assume $k^3 + 2k$ is divisible by 3. That is, we have
    $$k^3 + 2k = 3q \tag{H}$$
    for some polynomial $q(x)$.
    At case $n = k+1$, we have $(k+1)^3 + 2(k+1)$, so
    $$\begin{aligned}
    (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
    &= k^3 + 2k + 3k^2 + 3k + 3 \\
    &= k^3 + 2k + 3(k^2 + k + 1) \\
    &\overset{(H)}{=} 3q + 3(k^2 + k + 1) \\
    &= 3(q + k^2 + k + 1)
    \end{aligned}$$
    Therefore $(k+1)^3 + 2(k+1)$ is divisible by 3.

11. Base case: for $n = 0$ we have $17 \cdot 1^3 + 103 \cdot 1 = 120 = 6 \cdot 20$
Induction hypothesis: assume for some k, the term $17k^3 + 103k$ is divisible by 6. That is, we have

$$17k^3 + 103k = 6q \tag{H}$$

for some positive integer q.

At case $n = k+1$, we have $17(k+1)^3 + 103(k+1)$, so

$$\begin{aligned} 17(k+1)^3 + 103(k+1) &= 17(k^3 + 3k^2 + 3k + 1) + 103k + 103 \\ &= 17k^3 + 103k + 17 \cdot 3k^2 + 17 \cdot 3k + 17 + 103 \\ &\overset{(H)}{=} 6q + 3 \cdot 17k^2 + 3 \cdot 17k + 120 \\ &= 6q + 3(17k^2 + 17k) + 6 \cdot 20 \\ &= 6(q+20) + 3 \cdot 17k(k+1) \end{aligned}$$

Note that $k(k+1)$ is even, that is, $k(k+1)$ can be written as $2m$ for some positive integer m. Therefore we have $3 \cdot 17k(k+1) = 3 \cdot 17 \cdot 2m = 6 \cdot 17m$ and thus $6(q+20) + 3 \cdot 17k(k+1) = 6(q+20) + 6 \cdot 17m = 6(q+20+17m)$. Therefore $17(k+1)^3 + 103(k+1)$ is divisible by 6.

12. Base case: for $n = 0$ we have $5^{2 \cdot 0+1} + 2^{2 \cdot 0+1} = 7 = 7 \cdot 1$
Induction hypothesis: assume for some k, the term $5^{2k+1} + 2^{2k+1}$ is divisible by 7. That is, we have

$$5^{2k+1} + 2^{2k+1} = 7q \tag{H}$$

for some positive integer q.

At case $n = k+1$, we have $5^{2(k+1)+1} + 2^{2(k+1)+1} = 5^{2k+2+1} + 2^{2k+2+1} = 5^{2k+1+2} + 2^{2k+1+2} = 5^{2k+1} \cdot 25 + 2^{2k+1} \cdot 4$, so

$$\begin{aligned} 5^{2(k+1)+1} + 2^{2(k+1)+1} &= 5^{2k+1} \cdot (21+4) + 2^{2k+1} \cdot 4 \\ &= 5^{2k+1} \cdot 21 + 5^{2k+1} \cdot 4 + 2^{2k+1} \cdot 4 \\ &= 5^{2k+1} \cdot 21 + (5^{2k+1} + 2^{2k+1})4 \\ &\overset{(H)}{=} 5^{2k+1} \cdot 21 + (7q)4 \\ &= 5^{2k+1} \cdot 3 \cdot 7 + (7q)4 \\ &= (5^{2k+1} \cdot 3 + 4q) \cdot 7 \end{aligned}$$

So $5^{2(k+1)+1} + 2^{2(k+1)+1}$ is divisible by 7.

13. Base case: for $n = 1$ we have $2^1 + 1 = 3$ is divisible by 3.
Induction hypothesis: assume for some odd k, the term $2^k + 1$ is divisible by 8. That is, we have

$$2^k + 1 = 3q \tag{H}$$

for some positive integer q.

At case $n = k+2$ (not $k+1$ because k is odd and the next odd number is $k+2$), we have

$$2^{k+2} + 1 \;=\; 2^k 4 + 1 \;=\; 2^k 4 + 1 + 3 - 3 \;=\; 2^k 4 + 4 - 3 \;=\; (2^k+1)4 - 3 \;\overset{H}{=}\; 3q4 - 3 \;=\; 3(4q-1)$$

So $2^{k+2} + 1$ is divisible by 3.

14. Base case: for $n = 1$ we have $1^2 - 1 = 0$ is divisible by 8 (zero is divisible by everything).
Induction hypothesis: assume for some odd k, the term $k^2 - 1$ is divisible by 8. That is, we have

$$k^2 - 1 = 8q \tag{H}$$

for some positive integer q.

At case $n = k+2$ (not $k+1$ because k is odd and the next odd number is $k+2$), we have

$$(k+2)^2 - 1 \;=\; k^2 + 4k + 4 - 1 \;=\; k^2 - 1 + 4(k+1) \;\overset{H}{=}\; 8q + 4(k+1)$$

Since $k > 1$ is odd, then $k+1$ is even. Therefore, $k+1$ can be written as $2m$ where m is an positive integer. Hence

$$(k+2)^2 - 1 \;=\; 8q + 4(k+1) \;=\; 8q + 4(2m) \;=\; 8q + 8m \;=\; 8(q+m)$$

So $(k+2)^2$ is divisible by 8.

15. Base case: $1^4 - 1 = 0$ is divisible by 16
Induction hypothesis: assume for some odd k, the term $k^4 - 1$ is divisible by 16. That is, we have

$$k^4 - 1 = 16q \tag{H}$$

for some positive integer q.

At case $n = k + 2$ (not $k + 1$ because k is odd and the next odd number is $k + 2$), we have

$$\begin{array}{rcl}(k+2)^4 - 1 &=& k^4 + 4k^3 2 + 6k^2 4 + 4k8 + 16 - 1 \\ &=& k^4 - 1 + 8k^3 + 24k^2 + 32k + 16 \\ &\overset{H}{=}& 16q + 8k^3 + 24k^2 + 16 \cdot 2k + 16 \\ &=& 16q + 8(k^3 + 3k^2 + 4k + 2) \\ &=& 16q + 8(k+1)(k^2 + 2k + 2)\end{array}$$

Since $k > 1$ is odd, then $k + 1$ is even. Therefore, $k + 1$ can be written as $2m$ where m is an positive integer. Hence

$$(k+2)^4 - 1 \;=\; 16q + 8(2m)(k^2 + 2k + 2) \;=\; 16(q + m(k^2 + 2k + 2))$$

So $(k+2)^4 - 1$ is divisible by 16.

16. Base case $n = 1$, $\displaystyle\sum_{i=0}^{1} r^i = 1 + r = \frac{1 - r^2}{1 - r}$

Induction hypothesis (H): assume $\displaystyle\sum_{i=0}^{k} r^i = \frac{1 - r^{k+1}}{1 - r}$

At case $n = k + 1$, we have

$$\begin{array}{rcl}\displaystyle\sum_{i=0}^{k+1} r^i &=& \displaystyle\sum_{i=0}^{k} r^i + r^{k+1} \\ &\overset{H}{=}& \dfrac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &=& \dfrac{1 - r^{k+1}}{1 - r} + r^{k+1}\dfrac{1 - r}{1 - r} \\ &=& \dfrac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &=& \dfrac{1 - r^{k+2}}{1 - r}\end{array}$$

17. Base case: $n = 1$ we have $S(1) = 1 = 1^2$.

Induction hypothesis (H): assume $S(k) = 1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

For $n = k + 1$ we have

$$\begin{array}{rcl}S(k+1) &=& 1 + 3 + 5 + \cdots + (2k-1) + 2(k+1) - 1 \\ &=& S(k) + 2k + 1 \\ &\overset{H}{=}& k^2 + 2k + 1 \\ &=& (k+1)^2\end{array}$$

Solution approach two Consider the first $m = 2n$ intergers

$$\begin{array}{rcl}T(n) = 1 + 2 + 3 + \cdots + m &=& (1 + 3 + 5 + \cdots + 2n - 1) + (2 + 4 + 6 + \cdots + 2n) \\ \dfrac{1}{2}\big(m^2 + m\big) &=& S(n) + 2(1 + 2 + 3 + \cdots + n) \\ &=& S(n) + n^2 + n\end{array}$$

So

$$S(n) = \frac{1}{2}\big(m^2 + m\big) - n^2 - n \;\overset{m=2n}{\Longrightarrow}\; S(n) = \frac{1}{2}\big(4n^2 + 2n\big) - n^2 - n \;=\; n^2.$$

18. Base case: $n = 1$ we have $S(1) = 3 + 5 = 8 = 2.5 + 5.5$.

Induction hypothesis (H): assume $S(k) = 8 + 13 + 18 + 23 + ... + (3 + 5n) = 2.5k^2 + 5.5k$.

For $n = k+1$ we have

$$\begin{array}{rcl} S(k+1) & = & 8+13+18+23+...+(3+5k)+(3+5(k+1))) \\ & = & S(k)+3+5k+5 \\ & \overset{H}{=} & 2.5k^2+5.5k+8+5k \\ & = & 2.5k^2+5.5k+2.5+5.5+2\times 2.5k \\ & = & 2.5k^2+2\times 2.5k+2.5+(5.5k+5.5) \\ & = & 2.5(k^2+2k+1)+5.5(k+1) \\ & = & 2.5(k+1)^2+5.5(k+1) \end{array}$$

19. Base case: $n = 1$ we have $1+x \geq 1+x$ because $\geq$ means $>$ OR $=$.

    Induction hypothesis (H): assume $(1+x)^k \geq 1+kx$

    For $n = k+1$, we have

$$\begin{array}{rcll} (1+x)^{k+1} & = & (1+x)(1+x)^k & \\ & \overset{H}{\geq} & (1+x)(1+kx) & \\ & = & 1+kx+x+kx^2 & \\ & = & 1+(k+1)x+kx^2 & \\ & \geq & 1+(k+1)x & \because k\geq 1, x^2 \geq 0 \end{array}$$

20. Base case: $n = 2$ we have $2^2+4(2) = 12 < 4^2 = 16$.

    Induction hypothesis (H): assume $k^2+4k < 4^k$ for some $k > 2$.

    For $n = k+1$, we want to show $(1+k)^2+4(k+1) < 4^{k+1}$, now (this is tricky), consider

$$\frac{(1+k)^2+4(k+1)}{k^2+4k} = \frac{1+2k+k^2+4k+4}{k^2+4k} = \frac{k^2+4k+2k+5}{k^2+4k} = 1+\frac{2k+5}{k^2+4k}$$

    Now for $k > 2$ we have $2k < k^2$ and $5 < 4k$ so $\dfrac{2k+5}{k^2+4k} < 1$ and therefore

$$\frac{(1+k)^2+4(k+1)}{k^2+4k} = 1+\frac{2k+5}{k^2+4k} < 1+1 < 4$$

    Hence $(1+k)^2+4(k+1) < 4(k^2+4k) \overset{H}{<} 4\cdot 4^k = 4^{k+1}$

21. Base case is true $\frac{1}{2} < 1$.

    Induction hypothesis (H): assume $\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^k} < 1$

    For $n = k+1$, we need some trick.

    First we look at what will not work.
    If we consider $\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^k}+\dfrac{1}{2^{k+1}}$, then by (H) we have

$$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} < 1+\frac{1}{2^{k+1}}.$$

    This approach will not work.

    Here to show the case $n = k+1$, we need a clever way to use (H), and here is the trick: consider

$$\begin{array}{rcl} \frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} & = & \frac{1}{2}\Big(1+\frac{1}{2^2}+\cdots+\frac{1}{2^{k-1}}+\frac{1}{2^k}\Big) \\ & \overset{H}{<} & \frac{1}{2}\Big(1\Big) = 1. \end{array}$$

22. Base case: $\sqrt{1} < 3$.

    Induction hypothesis (H): assume $\sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{k}}}} < 3$.

    For case $n = k+1$, we have

$$\sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{k+\sqrt{k+1}}}}}$$

Now we shift all the indices $\{2,3,...,k,k+1\}$ by 1 by using $\ell = k+1$, we have

$$\sqrt{1+\underbrace{\sqrt{1+\sqrt{2+\cdots+\sqrt{\ell-1+\sqrt{\ell}}}}}_{\ell=k+1 \text{ for indicies } 2,3,\dots,k+1}} \overset{H}{<} \sqrt{1+3} = 2 < 3.$$

23. Base case: $4! = 4(3)(2)(1) = 24 > 16 = 2^4$.

    Induction hypothesis (H): assume $k! > 2^k$.

    For case $n = k+1$, we have

$$\begin{aligned} (k+1)! &= (k+1)k \\ &\overset{H}{>} 2^k \cdot k \\ &\overset{k>2}{>} 2^k \cdot 2 \\ &= 2^{k+1} \end{aligned}$$

24. Base case: $1 - \dfrac{1}{2^2} = \dfrac{3}{4} = \dfrac{2+1}{2(2)}$.

    Induction hypothesis (H): assume $\displaystyle\prod_{i=2}^{k}\Big(1-\frac{1}{i^2}\Big) = \frac{k+1}{2k}$

    For case $n = k+1$, we have

$$\begin{aligned} \prod_{i=2}^{k+1}\Big(1-\frac{1}{i^2}\Big) &= \Big(1-\frac{1}{(k+1)^2}\Big)\prod_{i=2}^{k}\Big(1-\frac{1}{i^2}\Big) \\ &\overset{H}{=} \Big(1-\frac{1}{(k+1)^2}\Big)\frac{k+1}{2k} \\ &= \frac{(k+1)^2-1}{(k+1)^2}\cdot\frac{k+1}{2k} \\ &= \frac{k^2+2k}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)} \end{aligned}$$

25. Base case: $n = 2$. We have LHS $= \sqrt{2}$ and RHS $= 2 - \dfrac{1}{2}$ both being positive. To show LHS $<$ RHS, we consider showing LHS^2 $<$RHS^2. LHS$^2 = 2$, and RHS$^2 = 4 - 2 + \frac{1}{4}$. Clearly $2 < 2 + \dfrac{1}{4}$ since $\dfrac{1}{4} > 0$.

    Induction hypothesis (H): assume $k^{1/k} < 2 - \dfrac{1}{k}$ for some integer $k > 2$.

    For case $n = k+1$, LHS$= (k+1)^{\frac{1}{k+1}}$ and RHS$= 2 - \dfrac{1}{k+1}$. Now we do some analysis, suppose LHS$<$RHS, then

$$\begin{aligned} & (k+1)^{\frac{1}{k+1}} < 2 - \frac{1}{k+1} \\ \iff\quad & k+1 < (2-\frac{1}{k+1})^{k+1} \\ \iff\quad & k < (2-\frac{1}{k+1})^{k+1} - 1 \\ \iff\quad & k^{1/k} < \Big((2-\frac{1}{k+1})^{k+1}-1\Big)^{1/k} \\ \overset{(H)}{\iff}\quad & 2-\frac{1}{k} < \Big((2-\frac{1}{k+1})^{k+1}-1\Big)^{1/k} \\ \iff\quad & (2-\frac{1}{k})^k < (2-\frac{1}{k+1})^{k+1} - 1 \end{aligned}$$

    So if we can show the last inequality is true, we proved the case for $n = k+1$.

    Now we consider the following function

$$f(x) = \Big(2-\frac{1}{x+1}\Big)^{x+1} - \Big(2-\frac{1}{x}\Big)^{x} - 1$$

    By derivative test, $f(x) > 0$ so the statement is true.

26. Let $n \in \mathbb{N}$. Prove the binomial theorem using induction.

Base case Consider case $n=1$, we have $(x+y)^1 = x+y \;=\; \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$.
Hypothesis step Assume the statement at $n=m$

$$(x+y)^m \;=\; \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} \tag{$\mathcal{H}$}$$

Induction step Consider the statement at $m+1$, we have

$$\begin{aligned}
(x+y)^{m+1} \;&=\; (x+y)(x+y)^m \\
&\overset{(\mathcal{H})}{=} (x+y)\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} \\
&=\; (x+y)\Big(\tbinom{m}{0}x^0y^m + \tbinom{m}{1}x^1y^{m-1} + \cdots + \tbinom{m}{m-1}x^{m-1}y^1 + \tbinom{m}{m}x^my^0\Big) \\
&=\; \begin{array}{l}\tbinom{m}{0}x^1y^m + \tbinom{m}{1}x^2y^{m-1} + \cdots + \tbinom{m}{m-1}x^my^1 + \tbinom{m}{m}x^{m+1}y^0 \\ +\tbinom{m}{0}x^0y^{m+1} + \tbinom{m}{1}x^1y^m + \cdots + \tbinom{m}{m-1}x^{m-1}y^2 + \tbinom{m}{m}x^my^1\end{array} \\
&=\; \tbinom{m}{0}x^0y^{m+1} + \Big(\tbinom{m}{0} + \tbinom{m}{1}\Big)x^1y^m + \Big(\tbinom{m}{1} + \tbinom{m}{2}\Big)x^2y^{m-1} + \cdots + \Big(\tbinom{m}{m-1} + \tbinom{m}{m}\Big)x^my^1 + \tbinom{m}{m}x^{m+1}y^0 \\
&=\; \tbinom{m+1}{0}x^0y^{m+1} + \tbinom{m+1}{1}x^1y^m + \tbinom{m+1}{2}x^2y^{m-1} + \cdots + \tbinom{m}{m}x^my^1 + \tbinom{m+1}{m+1}x^{m+1}y^0 \\
&=\; \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k} \qquad \square
\end{aligned}$$

2 Combinatorics

2.1 Counting problems

1. Let $A = \{a, b, c, d\}$. Enumerate all the possible subsets with the following rules
   - S_1 is the set of all possible set that contains one element from A
     I.e., $S_1 = \{x \mid x \in A\}$
   - $S_2^{=}$ is the set of all possible set that contains two elements from A and possibly repetitive.
     I.e., $S_2 = \{xy \mid x \in A, y \in A\}$.
   - $S_2^{\neq}$ is the set of all possible set that contains two elements from A with no repetitive element.
     I.e., $S_2 = \{xy \mid x \in A, y \in A, x \neq y\}$.
   - C_2 is the set of all possible set that contains all the combination of two elements from A.
     I.e., $C_2 = \{xy \mid x \in A, y \in A, x \neq y, xy = yx\}$.
   - P_2 is the set of all possible set that contains all the permutations of two elements from A.
     I.e., $P_2 = \{xy \mid x \in A, y \in A, x \neq y, xy \neq yx\}$.
2. Let A be a 4-set (i.e., $A = \{a, b, c, d\}$). How many number of combinations are possible if you are allowed to select not more than 2 (including 2) elements from A?
3. Let A be a 5-set (i.e., $A = \{a, b, c, d, e\}$). How many number of combinations are possible if you are allowed to select not more than 3 (including 3) elements from A?
4. Let A be a n-set (i.e., $A = \{1, 2, , ..., n\}$). How many number of combinations are possible if you are allowed to select not more than k (including k) elements from A?
5. Let A be a 5-set (i.e., $A = \{a, b, c, d, e\}$).
   - How many strings of length-4 can be formed using letters in A if repetitions are not allowed?
   - How many strings of length-4 can be formed using letters in A if repetitions are not allowed and begin with the letter b?
   - How many strings of length-4 can be formed using letters in A if repetitions are not allowed and do not begin with the letter b?
6. Power set of a set X, denoted as 2^X is the set of all subset of X. How many element has 2^X if X has n elements?
7. How many subsets with 3 or more elements does a set with 100 elements have?
8. There are 18 apples and 12 oranges in a store. How many ways of selecting 1 fruit (either an apple or an orange)?
9. How many ways of selecting an integer less than 10 (inclusive) that is even or prime?
10. How many length-7 words begin with a vowel $\{a, e, i, o, u\}$ or end with a vowel?
11. In a multiple-choice exam, there are 8 questions and each question has 3 answer (1 correct and 2 wrong). How many number of ways of answering all the questions?
12. How many different 6-character passwords (of uppercase and digits only, no lowercase) are there if (a) the first 2 characters must be different uppercase letter and the next 4 must be different digits, (b) the first 2 characters must be different digits and the next 4 must be different uppercase letters?
13. How many ways can five men and five women queue in a line so that men and women alternate?
14. The number of 3-digit decimal numbers with no repeated digit (leading zeros allowed)
15. The number of 3-digit decimal numbers with repeated digit allowed (leading zeros allowed)
16. Ten different paintings are to be allocated to 14 rooms so that no room get more than 1 painting. Find the number of ways of accomplishing this.
17. Ten different paintings are to be allocated to 6 rooms so that no room get more than 1 painting. Find the number of ways of accomplishing this.
18. How many strings of length-3 over alphabet $\{a, e, i, o, u\}$ has no repeated letter?
19. If $A = \{1, 2\}$ and $B = \{a, b, c\}$, how many functions $f : A \to B$ are one-to-one (injective)?

20. If $|A| = 3$ and $|B| = 5$, how many functions $f : A \to B$ are one-to-one (injective)?
21. If $|A| = m$ and $|B| = n$, how many functions $f : A \to B$ are one-to-one (injective)?
22. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, how many functions $f : A \to B$ are on-to (surjective)?
23. How many permutations of ABCDEFGH contain the string "DEF" as a substring?
24. Seven people ABCDEFG in cinema. How many ways can they be seated in a row so that A and B are not next to each other?
25. $A = \{a, b, c\}$. How many ways of permuting elements from A?
26. There are 15 apples and 4 oranges. How many ways to pick four fruits if exactly one fruit is orange?
27. Consider the *circular* permutation of a set $S = \{A, B, C\}$ around a circle. For example, ABC is the same as BCA under circular permutation. Enumerate all distinct circular permutations of S.
28. Consider the permutation of a set $\{A, B, C, D\}$ around a circle. How many distinct circular permutation are there?
29. How many ways of flipping 20 coins will give the same number of heads and tails?
30. There are 10 English speakers, 15 French speakers and 20 Italian speakers. How many pairs can be arranged so that in each pair people (a) speak different language, (b) speak the same language?
31. You are constructing passwords. There are two set of characters: alphabet and digit. There are 26 alphabet $\{a, b, c, ..., z\}$ and 10 digits $\{0, 1, 2, ..., 9\}$. You pick 6 character to form a string (password). How many strings are possible with at least 1 character being repeated at least once?

2.2 Binomial, trinomial and multinomial problems

1. What is the expansion of $(x + y)^5$? How many terms are there?
2. What is the expansion of $(2x - y)^4$? How many terms are there?
3. What is the expansion of $(ax \mp by)^3$?
   (The symbol $p \mp q$ means consider two cases: $p - q$ and $p + q$.)
4. What is the coefficient of x^3y^3 in $(2x - 3y)^6$?
5. What is the coefficient of x^2y^5 in $(x - 2y)^7$?
6. What is the coefficient of x^2 in $\left(x - \dfrac{1}{2x}\right)^7$?
7. What is the coefficient of $\dfrac{1}{x^2}$ in the expansion of $\left(x - \dfrac{2}{x^2}\right)^{10}$?
8. Prove that there is no x^3 in $(1 - 2x^2)^4$.
9. Prove $\dbinom{2n}{2} = n^2 + 2\dbinom{n}{2}$ algebraically by showing LHS (left hand-side) = RHS (right hand-side).
10. Prove $\dbinom{2n}{3} = 2\dbinom{n}{3} + 2n\dbinom{n}{2}$ algebraically.
11. Prove $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$ by binomial theorem.
12. Prove $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ by binomial theorem.
13. Prove $\displaystyle\sum_{k=\text{even}}^{n} \binom{n}{k} = \sum_{k=\text{odd}}^{n} \binom{n}{k} = 2^{n-1}$ by binomial theorem.
14. Using the binomial theorem, show that any power-of-3, up to power n, is a sum of power-of-2, up to power n.
15. Compute trinomial coefficient $\dbinom{7}{3, 2, 2}$ and $\dbinom{9}{2, 3, 4}$

16. Prove $\sum_{i,j}^{n} \binom{n}{i, j, n-i-j} = 3^n$ by trinomial theorem.
17. What is the coefficient of (a)z^4, (b)z^6 and (c)z^8 in $(z^2 + z^3 + z^4)^3$?
18. (**Hard**) What is the coefficient of x^{27} in $(x^4 + x^5 + \cdots)^5$?
19. How many terms are there in the expansion of $(x + y + 2x)^8$?
20. How many terms are there in the expansion of $(x + y + 2z)^8$?
21. How many different 5-letter arrangements can be formed from the word APPLE?
22. There are 20 marbles of the same size but of different colors (2 red, 8 blue, 10 green). Find the number of ways to permute these 20 marbles.
23. What is the coefficient of x^2 in $(1 + x + x^2 + x^3)^7$?
24. What is the coefficient of x^3 in $(1 + 2x - 3x^2 + 4x^3)^7$?

2.3 Lattice paths and divisible numbers

1. Find the number of shortest lattice paths from (0,0) to (3,2).
2. Find the number of shortest lattice paths from (2,1) to (3,2).
3. Find the number of shortest lattice paths from (0,0) to (10,9) that pass through (1,1), and (2,2).
4. How many integers between 50 and 999 inclusive are divisible by 6 but not by 5?
5. How many integers between 50 and 999 inclusive are divisible by 6 or 9?

2.4 Generating function and recursion problems

1. A box contains many identical red, blue, white and green marbles. Find the generating function corresponding to the problem of finding the number of ways of choosing r marbles from the box such that the sample does not have more than 2 red, more than 3 blue, more than 4 white and more than 5 green.
2. Find the generating function that solves the problem of finding the number of 5-digit integers with digit sum r.
3. Find the number of solutions in integer of the equation $a + b + c + d = 17$, where $1 \le a \le 3$, $2 \le b \le 4$, $3 \le c \le 5$, $4 \le d \le 6$.
4. How many ways can 40 voters cast their 40 votes for 5 candidates such that non candidate gets more than 10 votes? Write down the corresponding generating polynomial and state which coefficient corresponds to the answer of this problem.
5. From $n = 0, 1, 2, 3, ..., f_n = 1, 1, 1, 1, ...$ Find its generating function $F(x)$
6. From $n = 0, 1, 2, 3, ..., f_n = 1, -1, 1, -1, ...$ Find its generating function $F(x)$
7. From $n = 0, 1, 2, 3, ..., f_n = 1, 2, 3, 4, ...$ Find its generating function $F(x)$
8. From $n = 0, 1, 2, 3, ..., f_n = 1, -2, 3, -4, ...$ Find its generating function $F(x)$
9. From $n = 0, 1, 2, 3, ..., f_n = 0, 1, 2, 3, ...$ Find its generating function $F(x)$
10. Determine the partial fraction of $\dfrac{6x+5}{(2x-1)^2}$
11. Find a closed form for the recurrence $f_n = \begin{cases} 0 & n = 0 \\ 3f_{n-1} & n \ge 1 \end{cases}$
12. Find a closed form for the recurrence $f_n = \begin{cases} 1 & n = 0 \\ 3f_{n-1} & n \ge 1 \end{cases}$
13. Find a closed form for the recurrence $f_n = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 4f_{n-2} & n \ge 2 \end{cases}$

14. Find a closed form for the recurrence $f_n = \begin{cases} 2 & n=0 \\ 4 & n=1 \\ 4f_{n-1} - 2f_n & n \geq 2 \end{cases}$. Hint: $1-4x+2x^2 = \big(1-(2+\sqrt{2})x\big)\big(1-(2-\sqrt{2})x\big)$

2.5 Pigeonhole principle problems

1. Ten people are swimming in the lake. By Pigeonhole principle, at least n of them were born on the same day of the week. What is the smallest n?
2. Suppose there are 202 people representing the whole UK population to vote for three people in UK Prime Minister election. What is the minimum number of votes needed for someone to win the election?
3. You write down random positive integers. Prove that if you write 1001 numbers, then there must be at least 2 with the same last three digits.
4. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Show that if S is any subset of X with 7 elements, then there are 2 elements of S whose sum is 10.
5. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$. Show that if S is any subset of X with 7 elements, then there are 2 elements of S whose sum is 12.
6. What is the minimum number of students we need to have such that there are at least 2 students whose last names begin with the same letter of the (English) alphabet?
7. A box has many red balls and blue balls. You draw balls. What is the smallest number of balls drawn that will guarantee
   - there are 2 of the same colour?
   - there are 3 of the same colour?
   - there are 11 of the same colour?
8. There are 10 people in a room. Some of them might know each other but some might not. Using pigeonhole principle, show that at least two of them know the same number of people in the room. State clearly what is the pigeonhole and what is the pigeon.
9. Let "first name initial" means the first letter of a given name. In a room, at least 6 people share the same first name initial. What is the minimum number of people needed to achieve this?
10. True or false: The Pigeonhole Principle tells us that if we have $n+1$ pigeons and n holes, since $n+1 > n$, each box will have at least one pigeon.
11. True or false: The Pigeonhole Principle tells us that with n pigeons and k holes each hole can have at most $\lceil n/k \rceil$ pigeons.

2.6 Memory check

1. What is the inclusion-exclusion principle?
2. What is the sum rule? What situation can you use the sum rule?
3. What is the product rule? What situation can you use the product rule?
4. What is the subtraction rule? What situation can you use the subtraction rule?
5. What is the division rule? What situation can you use the division rule?
6. What is the formula for choosing k combination (repetition not allowed) in a n-set?
7. What is the formula for choosing k combination (repetition allowed) in a n-set?
8. What is the formula for choosing k permutation (repetition not allowed) in a n-set?
9. What is the formula for choosing k permutation (repetition allowed) in a n-set?
10. What are $\lceil x \rceil$ and $\lfloor x \rfloor$?

2.7 Solution to counting problems

1. - $S_1 = \{a, b, c, d\}$
   - $S_2 = \{aa, ab, ac, ad, ba, bb, bc, bd, ca, cb, cc, cd, da, db, dc, dd\}$
   - $S_2^{\neq} = \{ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc\}$
   - $C_2 = \{ab, ac, ad, bc, bd, cd\}$
   - $P_2 = \{ab, ac, adba, bc, bd, ca, cb, cd, da, db, dc\}$
2. There are 11 ways to form combination by selecting not more than 2 elements from A.
   - Selecting 0 element: 1 way
   - Selecting 1 element: 4 ways
   - Selecting 2 elements: 6 ways

   Total: 1 + 4 + 6 = 11.
3. There are 26 ways to form combination by selecting not more than 2 elements from A.
   - Selecting 0 element: $\binom{5}{0} = 1$ way
   - Selecting 1 element: $\binom{5}{1} = 5$ ways
   - Selecting 2 elements: $\binom{5}{2} = 10$ ways
   - Selecting 3 elements: $\binom{5}{3} = 10$ ways

   Total: 1 + 5 + 10 + 10 = 26 ways.
4. $\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{k}$
5. Let A be a 5-set (i.e., $A = \{a, b, c, d, e\}$).
   - $5 \cdot 4 \cdot 3 \cdot 2 = 120$
   - $1 \cdot 4 \cdot 3 \cdot 2 = 24$
   - $120 - 24 = 96$
6. 2^n. Draw a tree to see it.
7. By complement: power set $\setminus$ $\{$set with less than 3 elements$\}$, which is

   $$\text{power set} \setminus \big(\{\text{set with 0 elements}\} \cup \{\text{set with 1 element}\} \cup \{\text{set with 2 elements}\}\big)$$

   We have $2^{100} - \left(\binom{100}{0} + \binom{100}{1} + \binom{100}{2}\right)$.
8. $|\text{apple} \cup \text{orange}| = |\text{apple}| + |\text{orange}| - \underbrace{|\text{apple} \cap \text{orange}|}_{=0} = 18 + 12$
9. $|\text{even} \cup \text{prime}| = |\text{even}| + |\text{prime}| - |\text{even} \cap \text{prime}|$.
   We have $|\{2, 4, 6, 8, 10\}| + |\{2, 3, 5, 7\}| - |\{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 7\}| = 5 + 4 - |\{2\}| = 9 - 1 = 8$.
10. $|\text{begin with vowel} \cup \text{end with vowel}| = |\text{begin with vowel}| + |\text{end with vowel}| - |\text{begin AND end with vowel}|$.
    We have $5 \cdot 26^6 + 26^6 \cdot 5 - 5 \cdot 26^5 \cdot 5$
11. By product rule, $3^8 = 6561$.
12. (a) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7$
    (b) $10 \cdot 9 \cdot 26 \cdot 25 \cdot 24 \cdot 23$
13. $5! = 120$ ways to arrange the men
    $5! = 120$ ways to arrange the women
    There are two ways to interlace the two: mwmwmwmwmw and wmwmwmwmwm
    Lastly by product rule, in total $120 \cdot 120 \cdot 2 = 28800$.

14. $10 \cdot 9 \cdot 8$.
We have three boxes $\square_1\square_2\square_3$ to fill.
$\square_1$ has 10 way to fill.
$\square_2$ has one less (to avoid repeated digit) ways to fill, i.e., $10 - 1 = 9$ ways.
$\square_2$ has two less (to avoid repeated digit) ways to fill, i.e., $10 - 2 = 8$ ways.
Lastly by product rule: $10 \cdot 9 \cdot 8$

15. 10^3

16. More room than painting, so 14-choose-10 ways to select 10 paintings among the 14 rooms.
Solution 1
For the 10 paintings there are $10!$ permutations, so $\binom{14}{10}10!$, this number is denoted as $P(14, 10)$ in some books.
Solution 2
Note 1: "no allocation" is also an "allocation", you can distribute a room with zero painting
Note 2: you don't need to use up all paintings
Let the person who distribute the painting be a janitor. For the 10 paintings, the janitor can

    - keep all the paintings and do not distribute to any room.
      "the janitor is selfish"
    - keep 9 paintings and only distribute 1 painting among all the rooms.
      "the janitor is extremely selfish"
    - keep 8 paintings and only distribute 2 paintings among all the rooms.
      "the janitor is very selfish"
    - and so on
    - keep 1 paintings and only distribute 9 paintings among all the rooms.
      "the janitor is a little bit selfish"
    - distribute all the 10 paintings among all the rooms.
      "the janitor is not selfish"
      This is the solution 1

    In the original question, there is no information regarding the janitor is selfish or not so if we assume the janitor is not selfish, then we have Solution 1. Other wise, we have

$$|\text{keep 10 distribute 0}| + |\text{keep 9 distribute 1}| + \cdots + |\text{keep 0 distribute 10}|$$

    which is

$$\underbrace{\binom{14}{0}0!}_{\text{keep 10 distribute 0}} \;+\; \underbrace{\binom{14}{1}1!}_{\text{keep 9 distribute 1}} \;+\; \underbrace{\binom{14}{2}2!}_{\text{keep 8 distribute 2}} \;+\cdots+\; \underbrace{\binom{14}{10}10!}_{\text{keep 0 distribute 10}}$$

    or equivalently

$$P(14, 0) + P(14, 1) + \cdots + P(14, 10).$$

    The issue here is that the question did not provide enough information, leading to this vagueness and therefore we have multiple solutions. The key point here is that

    - if you assumed the janitor is selfish then you have to get the solution $P(14, 0) + P(14, 1) + \cdots + P(14, 10)$.
    - if you assumed the janitor is not selfish then you have to get the solution $P(14, 10)$

17. More painting than rooms, so 10-choose-6 ways to select 6 rooms among the 10 paintings.
Solution 1 (janitor not selfish)
For the 6 rooms there are $6!$ permutations, so $\binom{10}{6}6!$, this number is denoted as $P(10, 6)$ in some books.
Solution 2 (janitor selfish)

$$\underbrace{\binom{6}{0}}_{\text{choose 0 room}} \;+\; \underbrace{\binom{6}{1}\binom{10}{1}0!}_{\text{distribute 1 among 1 rooms}} \;+\; \underbrace{\binom{6}{2}\binom{10}{2}2!}_{\text{distribute 2 among 2 rooms}} \;+\cdots+\; \underbrace{\binom{6}{6}\binom{10}{6}6!}_{\text{distribute 6 among 6 rooms}}$$

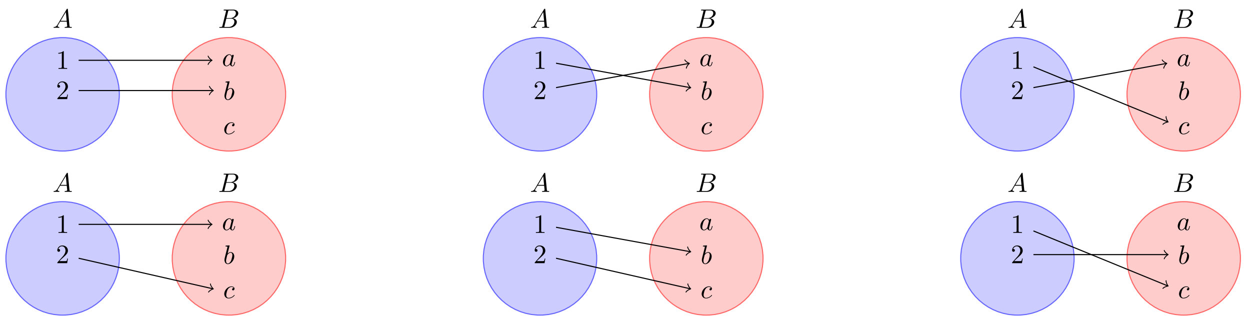
    Here

$$\underbrace{\binom{6}{k}}_{\text{pick } k \text{ rooms among 6 rooms}} \quad\cdot\quad \underbrace{\binom{10}{j}j!}_{\text{distribute } j \text{ paintings among 10 paintings}}$$

    Note: In exam, solution 1 is sufficient.

18. 5-choose-3 ways to select 3 letters from a 5-set, and each 3-letter group has 3! permutations, so $\binom{5}{3}3!$

19. $A = \{1, 2\}$ and $B = \{a, b, c\}$ has $\binom{3}{2}2! = 6$ injective maps.



Make sure you fully understand the condition of what is a function: every $a \in A$ has exactly one arrow pointing to one $b \in B$. If some $a \in A$ has no arrow pointing to B, such thing is not a function.

20. 5-choose-3 ways to point 3 arrows from A to B, and each arrow group has 3! permutations, $\binom{5}{3}3!$ See this by drawing a function map.

21. $\binom{n}{m}m!$

22. We need to recall two things
   - the definition of function
   - the definition of surjective function

Recall the definition of function

$$f : A \to B \text{ is a function if } \forall a \in A \text{ maps to } \textbf{exactly 1} \text{ element in } B.$$

Recall surjective map

$$f : A \to B \text{ is surjective if } \forall b \in B, \exists a \in A \,\text{s.t.}\, f(a) = b.$$

We can have

- One single $a \in A$ maps to several different $b \in B$
- Several different $a \in A$ maps to the same single $b \in B$

We cannot have

- One single $a \in A$ maps to nothing
- One single $b \in B$ not pointed by an arrow from A

Solution approach 1 From slide in 8 `https://angms.science/doc/COMP1215/COMP1215_Combinatorics.pdf`, the number of surjections, S_{np}, between a n-set A and a p-set B, where $n \geq p$, is

$$S_{np} = p^n - \binom{p}{1}(p-1)^n + \binom{p}{2}(p-2)^n + \cdots + (-1)^{p-1}\binom{p}{p-1} = \sum_{k=0}^{p}(-1)^k\binom{p}{k}(p-k)^n.$$

For $n = 3, p = 2$ we have

$$S_{3,2} \;=\; 2^3 - \binom{2}{1}(2-1)^3 + \binom{2}{2}(2-2)^3 \;=\; 8 - 2 = 6.$$

So there are 6 surjective maps from A to B.

Solution approach 2 The condition that "every $b \in B$ has to be mapped" means that we can think of the problem as distributing elements in A into $|B|$ disjoint sets.

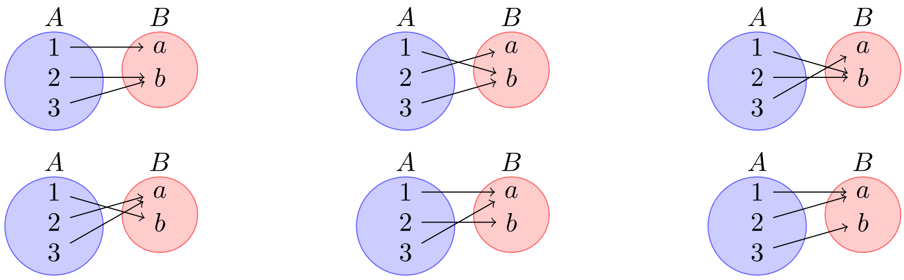
Here $B = \{a, b\}$ has $|B| = 2$ elements, so that means there are two boxes.

$$\underbrace{\{\}}_{\text{box for } a}, \underbrace{\{\}}_{\text{box for } b}.$$

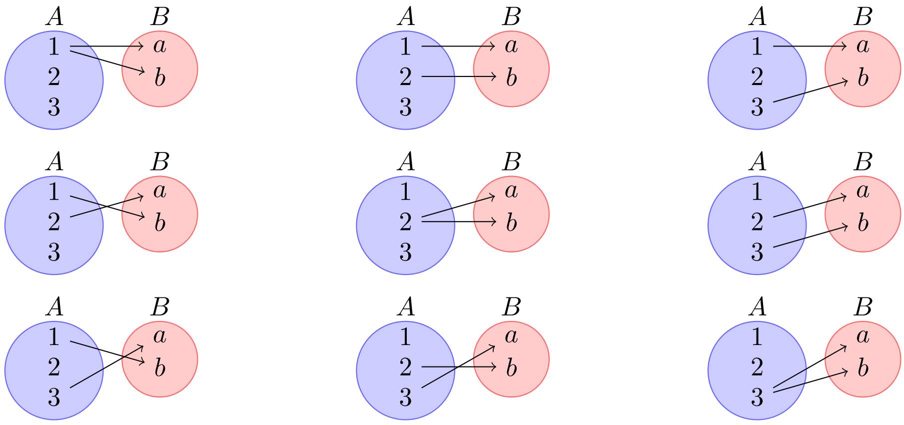
We are distributing $A = \{1, 2, 3\}$ into these two boxes that each box has at least one elements.

$$\begin{array}{c} \{1\}, \{2, 3\} \\ \{2\}, \{1, 3\} \\ \{3\}, \{1, 2\} \\ \{2, 3\}, \{1\} \\ \{1, 3\}, \{2\} \\ \{1, 2\}, \{3\} \end{array}$$

Solution approach 3: brute force / picture



Remark: the followings are NOT function (because null is not accepted)



23. Treat DEF as a single letter, and count permutations of ABC(DEF)GH. There are $6! = 720$ of these.

24. (#ways 7 people seat) $-$ (#ways AB seat next to each other) $= 7! - 2 \cdot 6! = 3600$

    Long explanation:

    A,B seat next to each other $\iff$ we see AB as a conjoined twins and therefore share the same-body as a single entity $\implies$ there is only 6 people

    A,B seat next to each other $\iff$ A on B's left OR A on B's right $\implies$ two ways

25. $\dfrac{3!}{1!} = 6$. They are ab, ba, ac, ca, bc, cb.

26. $\dbinom{4}{1}\dbinom{15}{3}$

    Long explanation:

    We pick 4 fruits = we have four boxes $\square\square\square\square$

    The $\dbinom{4}{1}$ corresponds to filling the four boxes with one orange o $\begin{cases} o\square\square\square \\ \square o\square\square \\ \square\square o\square \\ \square\square\square o \end{cases}$

    Notice that we do not label the oranges as $O = \{o_A, o_B, o_C, o_D\}$ because we do not distinguish between the four oranges. In other words, in principle we have

$$\underbrace{\begin{cases} o_A\square\square\square \\ \square o_A\square\square \\ \square\square o_A\square \\ \square\square\square o_A \end{cases}}_{o_A \text{ is selected}} \quad \underbrace{\begin{cases} o_B\square\square\square \\ \square o_B\square\square \\ \square\square o_B\square \\ \square\square\square o_B \end{cases}}_{o_B \text{ is selected}} \quad \underbrace{\begin{cases} o_C\square\square\square \\ \square o_C\square\square \\ \square\square o_C\square \\ \square\square\square o_C \end{cases}}_{o_C \text{ is selected}} \quad \underbrace{\begin{cases} o_D\square\square\square \\ \square o_D\square\square \\ \square\square o_D\square \\ \square\square\square o_D \end{cases}}_{o_D \text{ is selected}}$$

but we cannot distinguish between the four oranges (think of the orange is chosen by a robot arm in a dark room, and you cannot see what is happening in the dark room), so the above four cases are identical.

Now for the part $\binom{15}{3}$

We have 15 apples, we have 3 boxes, so it is 15-choose-3.

27. First we list all permutation and manually cycle them

$$ABC,\ ACB,\ \underbrace{BAC}_{=ACB},\ \underbrace{BCA}_{=ABC},\ \underbrace{CAB}_{=BCA=ABC},\ \underbrace{CBA}_{=ACB=BAC}. \tag{$*$}$$

28. $(4-1)!$.
How to see this: regular permutation is on a straight line. Circular permutation connects the head and tail into a circle, this removed one slot in the permutation. I.e., similar to Q13 that treating DEF as a single string, we are now treating the head-tail as a single entry, this procedure reduces the number of slot by 1.

29. $\binom{20}{10}$.
To see this, first we list all the possible number of (head,tails).

$$(0,20),\ (1,19),\ (2,18),\ \dots,\ (9,11),\ (10,10),\ (11,9),\ \dots,\ (20,0)$$

Since we are asking for same number of heads and tails, the only possible case is $(10,10)$, therefore, we are looking at how many ways of flipping 20 coins gives 10 heads (or 10 tails).

30. Let B = British, F = French and I = Italian.

(a)

- **Method 1 (direct approach)** We have $(B\times F)\cup(B\times I)\cup(F\times I)$ as the set that satisfies the requirement. Its cardinality is $|B\times F|+|B\times I|+|F\times I| = 10\cdot 15+10\times 20+15\times 20 = 650$
- **Method 2 (by complement)**

$$\underbrace{\binom{10+15+20}{2}}_{\text{all the ways to form a pair}} - \underbrace{\left(\binom{10}{2}+\binom{15}{2}+\binom{20}{2}\right)}_{\text{all the ways of forming same-language pairs}}$$

(b) Note that there is an ambiguity here. Suppose we have a pair (X,Y), the question didn't give enough information for us to tell whether $(X,Y)=(Y,X)$ or $(X,Y)\neq(Y,X)$

- **Solution 1 (if we take $(X,Y)\neq(Y,X)$)**
  - **Method 1** The set $B\times B$ refers to the all the possibility of picking two English speakers (repetition allowed) from the group of English speakers, and $|B\times B|$ is the number of pair of English speakers (repetition allowed). Tricky part: a person cannot be selected twice (you cannot form a pair from one person), so the correct set should be $|B\times B|-|B|$.
    Similarly we have $|F\times F|-|F|$ and $|I\times I|-|I|$ for French and Italian speakers.
    So the total number of pairs that speak the same language is $10^2-10+15^2-15+20^2-20$.
  - **Method 2** For each pair of English speaker, we have 10 choice for selecting the first person in the pair, then the we have 9 choice for selecting the second person in the pair, so the number of English speaking pair is $10\cdot 9$. the total number of pairs that speak the same language is $10\cdot 9+15\cdot 14+20\cdot 19$.
- **Solution 2 (if we take $(X,Y)=(Y,X)$)**
  - **Method 1** $\binom{10}{2}+\binom{15}{2}+\binom{20}{2}$
  - **Method 2** The value of **Solution 1** $\times\dfrac{1}{2}$

31. Use set and complement.

$$\Big\{\text{6-string with at least one symbol repeated once}\Big\} = \Big\{\text{6-string}\Big\} \setminus \Big\{\text{6-string with all symbol unique }\Big\}$$

Subtraction rule: $|\{\text{6-string}\}| - |\{\text{6-string with all symbol unique }\}| = (26+10)^6 - \underbrace{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}_{P_6^{36}}$.

The answer is $36^6 - P_6^{36}$.

2.8 Solution to Binomial, trinomial and multinomial problems

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.
2. $(2x-y)^4 = \binom{4}{0}(2x)^4(-y)^0 + \binom{4}{1}(2x)^3(-y)^1 + \binom{4}{2}(2x)^2(-y)^2 + \binom{4}{3}(2x)^1(-y)^3 + \binom{4}{4}(2x)^0(-y)^4$
$= (2x)^4(-y)^0 + 4(2x)^3(-y)^1 + 6(2x)^2(-y)^2 + 4(2x)^1(-y)^3 + (2x)^0(-y)^4$
$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$
3. $a^3x^3 \mp a^2bx^2y + ab^2xy^2 \mp b^3y^3$
4. $\binom{6}{3}(2x)^3(-3y)^3 = -4320x^3y^3$, the coefficient is -4320
5. $\binom{7}{2}(-2)^5 = -672$
6. The kth term in the expansion is

$$\binom{7}{k}x^k\Big(-\frac{1}{2x}\Big)^{7-k} = \binom{7}{k}x^k\Big(-\frac{1}{2}\Big)^{7-k}\Big(\frac{1}{x}\Big)^{7-k} = \binom{7}{k}\Big(-\frac{1}{2}\Big)^{7-k}x^{2k-7}.$$

To get the coefficient of x^2 we need to find the term such that $2k-7=2$.

$$2k-7=2 \iff 2k=5 \iff \text{no integer solution for } k \iff \text{no solution} \iff [x^2]\Big(x-\frac{1}{2x}\Big)^7 = 0.$$

7. The kth term in the expansion is

$$\binom{10}{k}x^k\Big(-\frac{2}{x^2}\Big)^{10-k} = \binom{10}{k}x^k(-2)^{10-k}x^{2k-20} = \binom{10}{k}(-2)^{10-k}x^{3k-20}$$

To get the coefficient of x^{-2} we need to find the term such that $3k-20=-2$, which gives $k=6$, so the answer is $\binom{10}{6}2^4$.

8. The kth term in the expansion is $\binom{4}{k}(-2x^2)^k = \binom{4}{k}(-2)^kx^{2k}$. There is no integer k such that $x^{2k}=x^3$.
9. LHS$=\binom{2n}{2} = \dfrac{(2n)!}{(2n-2)!2!} = \dfrac{(2n)(2n-1)(2n-2)!}{(2n-2)!2} = n(2n-1) = 2n^2-n$
RHS$= n^2 + 2\binom{n}{2} = n^2 + 2\dfrac{(n)!}{(n-2)!2!} = n^2 + 2\dfrac{n(n-1)}{2} = n^2 + n(n-1) = 2n^2-n =$LHS
10. LHS$=\dfrac{2n(2n-1)(2n-2)}{6} = \dfrac{4n^3-6n+2}{3}$
RHS$= 2\dfrac{n(n-1)(n-2)}{6} + 2n\dfrac{n(n-1)}{2} = \dfrac{n^3-3n^2+2}{3} + n^3 - n^2 = \dfrac{n^3-3n^2+2}{3} + \dfrac{3n^3-3n^2}{3} = \dfrac{4n^3-6n+2}{3}$ =LHS
11. $2^n = (1+1)^2 = \sum_{k=0}^{n}\binom{n}{k}1^k1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}$.
12. $0^n = (-1+1)^2 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$.
13. Add $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ and $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ gives the result.

14. $3^n = (2+1)^2 = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$

15. $\dbinom{7}{3,2,2} = \dfrac{7!}{3!2!2!} = 210$, $\dbinom{9}{2,3,4} = \dfrac{9!}{2!3!4!} = 1260$

16. $3^n = (1+1+1)^2 = \displaystyle\sum_{i,j}^{n} \binom{n}{i,j,n-i-j} 1^i 1^j 1^{n-i-j} = \sum_{i,j}^{n} \binom{n}{i,j,n-i-j}.$

17. $(z^2+z^3+z^4)^3 = \Big(z^2(1+z+z^2)\Big)^3 = z^6(1+z+z^2)^3$, so for (a) it is 0, for (b) it is 1. For (c) it is the coefficient of z^2 in $(1+z+x^2)^3$. In the trinomial expansion, we have

$$\binom{3}{i,j,3-i-j} z^i (z^2)^j 1^{3-i-j} \;=\; \frac{3!}{i!j!(3-i-j)!} z^i z^{2j} 1^{3-i-j} \;=\; \frac{3!}{i!j!k!} z^{i+2j} 1^k$$

Because we now want the z^2 term, so $2 = i+2j$ with $i+j+k=3$ gives $(i,j,k) = (2,0,1)$ or $(0,1,2)$. Therefore

$$\frac{3!}{2!0!1!} + \frac{3!}{0!1!2!} = 3+3 = 6.$$

$(z^2+z^3+z^4)^3$ contains $6x^8$.

18. The coefficient of x^{27} in $(x^4+x^5+\cdots)^5$

$$[x^{27}]\Big(x^4+x^5+\cdots\Big)^5 \;=\; [x^{27}]\Big(x^4(1+x^1+\cdots)\Big)^5 \;=\; [x^{27}]x^{20}(1+x^1+\cdots)^5 \;=\; [x^7](1+x^1+\cdots)^5$$

So the coefficient of x^{27} in $(x^4+x^5+\cdots)^5$. is the same as the coefficient of x^7 in $(1+x+x^2+\cdots)^5$.
To find the coefficient of x^7 in $(1+x+x^2+\cdots)^5$, we need to use the following in slide 71/90 of the slide

$$(1+x+x^2+\cdots)^n = \sum_{i=1}^{\infty} \binom{n+r-1}{r} x^r$$

so the coefficient of x^7 in $(1+x+x^2+\cdots)^5$ with $n=5, r=7$

$$\binom{5+7-1}{7} = \binom{11}{7} = \frac{11!}{7!4!}.$$

19. $(x+y+2x)^8 = (3x+y)^8$ has 9 terms

$$\binom{8}{0}(3x)^8y^0+\binom{8}{1}(3x)^7y^1+\binom{8}{2}(3x)^6y^2+\binom{8}{3}(3x)^5y^3+\binom{8}{4}(3x)^4y^4+\binom{8}{5}(3x)^3y^5+\binom{8}{6}(3x)^2y^6+\binom{8}{7}(3x)^1y^7+\binom{8}{8}(3x)^0y^8$$

20. Number of ways to choose 3 elements with replacement in a 8-set: $\dbinom{8+3-1}{3} = \dbinom{8+2}{2} = \dfrac{(8+1)(8+2)}{2} = 45$

21. $\dbinom{5}{1,2,1,1} = \dfrac{5!}{1!2!1!1!}$

22. $\dbinom{20}{2,8,10} = \dfrac{20!}{2!8!10!}$

23. To find the coefficient of x^2 in $(1+x+x^2+x^3)^7$, we consider the multinomial expansion of $(1+x+x^2+x^3)^7$:

$$(1+x+x^2+x^3)^7 = \sum_{\substack{0\le p,q,r,s\le 7\\ p+q+r+s=7}} \binom{7}{p,q,r,s} 1^p x^q (x^2)^r (x^3)^s = \sum_{\substack{0\le p,q,r,s\le 7\\ p+q+r+s=7}} \binom{7}{p,q,r,s} 1^p x^{q+2r+3s}$$

To find the coefficient of x^2 in this expansion, we look for the terms $x^{q+2r+3s}$ that gives x^2, that means we look for $q+2r+3s=2$ for all $0 \le p,q,r,s \le 7$. We have the following two choices for (p,q,r,s):

$$(5,2,0,0)$$
$$(6,0,1,0)$$

Hence,

$$[x^2](1+x+x^2+x^3)^7 \;=\; \binom{7}{5,2,0,0} + \binom{7}{6,0,1,0} \;=\; \frac{7!}{5!2!0!0!} + \frac{7!}{6!0!1!0!} \;=\; 21+7 \;=\; 28.$$

24. To find the coefficient of x^3 in $(1+2x-3x^2+4x^3)^7$ we consider the multinomial expansion of $(1+2x-3x^2+4x^3)^7$

$$\begin{aligned}(1+2x-3x^2+4x^3)^7 &= \sum_{\substack{0\le p,q,r,s\le 7\\ p+q+r+s=7}} \binom{7}{p,q,r,s} 1^p(2x)^q(-3x^2)^r(4x^3)^s \\ &= \sum_{\substack{0\le p,q,r,s\le 7\\ p+q+r+s=7}} \binom{7}{p,q,r,s} 1^p 2^q (-3)^r 4^s x^{q+2r+3s}\end{aligned}$$

To get the term x^3, we look for $q+2r+3s=3$ for $0\le p,q,r,s\le 7$. This give for (p,q,r,s)

$$\begin{array}{c}(4,3,0,0)\\(5,1,1,0)\\(6,0,0,1)\end{array}$$

Hence

$$\begin{aligned}[x^3](1+2x-3x^2+4x^3)^7 &= \binom{7}{4,3,0,0}1^4 2^3(-3)^0 4^0 + \binom{7}{5,1,1,0}1^5 2^1(-3)^1 4^0 + \binom{7}{6,0,0,1}1^6 2^0(-3)^0 4^1 \\ &= \frac{7!}{4!3!0!0!}8 - \frac{7!}{5!1!1!0!}6 + \frac{7!}{6!0!0!1!}4 \\ &= 280-252+28 \\ &= 56\end{aligned}$$

2.9 Solution to lattice paths and divisible numbers

1. $\binom{5}{3}$
2. $(2,1)\to(3,2)$ is the same as $(0,0)\to(1,1)$
   $\binom{5}{3}-\binom{3}{2}$
3. From $(0,0)$ to $(1,1)$ there are 2 shortest paths, from $(1,1)$ to $(2,2)$ there are 2 shortest paths, so by product rule, from $(0,0)$ to $(2,2)$ there are 4 shortest paths. From $(2,2)$ to $(10,9)$ is the same as from $(0,0)$ to $(8,7)$ and there are $\binom{15}{8}$ shortest paths. So the total number of shortest paths, by product rule is $4\binom{15}{8}$.
4. $\lfloor\frac{999}{6}\rfloor-\lfloor\frac{49}{6}\rfloor-\left(\lfloor\frac{999}{30}\rfloor-\lfloor\frac{49}{30}\rfloor\right)=126$
5. $\left(\lfloor\frac{999}{6}\rfloor-\lfloor\frac{49}{6}\rfloor\right)+\left(\lfloor\frac{999}{9}\rfloor-\lfloor\frac{49}{9}\rfloor\right)-\left(\lfloor\frac{999}{18}\rfloor-\lfloor\frac{49}{18}\rfloor\right)=211$

2.10 Solution to generating function and recursion problems

1. $(1+x+x^2)(1+x+x^2+x^3)(1+x+\cdots+x^4)(1+x+\cdots+x^5)$
2. The first digit is at least 1 and most 9; the other 4 digits are in 0 to 9. $(x+x^2+\cdots+x^9)(1+x+x^2+\cdots+x^9)^4$
3. The solution is the coefficient of x^{17} in $(x+x^2+x^3)(x^2+x^3+x^4)(x^3+x^4+x^5)(x^4+x^5+x^6)$.

$$\begin{aligned} & (x+x^2+x^3)(x^2+x^3+x^4)(x^3+x^4+x^5)(x^4+x^5+x^6) \\ = & \; x(1+x+x^2)x^2(1+x+x^2)x^3(1+x+x^2)x^4(1+x+x^2) \\ = & \; x^{10}(1+x+x^2)(1+x+x^2)(1+x+x^2)(1+x+x^2)\end{aligned}$$

So we now consider the coefficient of x^7 in $(1+x+x^2)^4$

$$\binom{4}{k,j}x^k(x^2)^j = \frac{4!}{k!j!(4-k-j)!}x^{k+2j}$$

For $k+2j=7$ with $k+j\le 4$ we have $(1,3)$

$$\frac{4!}{1!3!(4-1-3)!}=4.$$

There are 4 solutions.

4. Note that the problem is the same as finding the number of integer solutions to $a+b+c+d+e = 50$ with $0 \leq a, b, c, d, e \leq 10$. The generating function is

$$\underbrace{(1+x+x^2+\cdots+x^{10})}_{\text{voting pattern for candidate 1}}\underbrace{(1+x+x^2+\cdots+x^{10})}_{\text{voting pattern for candidate 2}}\cdots\underbrace{(1+x+x^2+\cdots+x^{10})}_{\text{voting pattern for candidate 5}} = (1+x+x^2+\cdots+x^{10})^5.$$

The coefficient of x^{40} of $(1+x+x^2+\cdots+x^{10})^5$ is the solution to this problem.

5. $F(x) = 1 + x + x^2 + x^3 + ... = \dfrac{1}{1-x}$

6. $F(x) = 1 - x + x^2 - x^3 + ... = \dfrac{1}{1+x}$

7. $F(x) = 1 + 2x + 3x^2 + 4x^3 + ... = \dfrac{1}{(1-x)^2}$

8. $F(x) = 1 - 2x + 3x^2 - 4x^3 + ... = \dfrac{1}{(1+x)^2}$

9. $F(x) = x + 2x^2 + 3x^3 + ... = \dfrac{x}{(1-x)^2}$

10. $\dfrac{6x+5}{(2x-1)^2} = \dfrac{A}{2x-1} + \dfrac{B}{(2x-1)^2} \iff 6x+5 = A(2x-1)+B$. Put $x = 0.5$ gives $B = 8$. Put $x = 0$ with $B = 8$ gives $A = 3$. So $\dfrac{6x+5}{(2x-1)^2} = \dfrac{3}{2x-1} + \dfrac{8}{(2x-1)^2}$.

11. $f_n = 0$ for all n. Actually you do not need to do anything. The base case is zero, multiplying zero gives zero.

12. $f_n = 3^n$.
How: let $F(x) = 1 + x + x^2 + \cdots$. Consider $F(x) - 3xF(x)$. Using $f_n = 3f_{n-1}$ by the definition of the recurrence gives $F(x) = \dfrac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots$.

13. Let $F(x) = 1 + x + x^2 + \cdots$. Consider $F(x) - 4x^2F(x) = 1 + x$ which gives $F(x) = \dfrac{1+x}{1-4x^2} = \dfrac{1+x}{(1-2x)(1+2x)}$ By partial fraction $F(x) = \dfrac{3}{4}\dfrac{1}{1-2x} + \dfrac{1}{4}\dfrac{1}{1-(-2)x}$. Therefore $f_n = \dfrac{3}{4} \cdot 2^n + \dfrac{1}{4} \cdot (-2)^n = \dfrac{3 \cdot 2^n + (-2)^n}{4}$

14. Let $F(x) = 1 + x + x^2 + \cdots$.

$$\begin{aligned}
F(x) &= f_0 + f_1x + f_2x^2 + \cdots \\
-4xF(x) &= -4f_0x - 4f_1x^2 - \cdots \\
2x^2F(x) &= 2f_0x^2 + \cdots \\
(1-4x+2x^2)F(x) &= f_0 + (f_1 - 4f_0)x + (f_2 - 4f_1 + 2f_0)x^2 + \cdots \\
&= 2 + (4 - 4(2))x + 0 + 0 \cdots \\
F(x) &= \frac{2-4x}{1-4x+2x^2} \\
&= \frac{A}{1-(2+\sqrt{2})x} + \frac{B}{1-(2-\sqrt{2})x} \\
&= \frac{1}{1-(2+\sqrt{2})x} + \frac{1}{1-(2-\sqrt{2})x} \\
&= \Big(1 + (2+\sqrt{2})x + (2+\sqrt{2})^2x^2 + \cdots\Big) + \Big(1 + (2-\sqrt{2})x + (2-\sqrt{2})^2x^2 + \cdots\Big) \\
f_n &= (2+\sqrt{2})^n + (2-\sqrt{2})^n
\end{aligned}$$

2.11 Solution to pigeonhole principle problems

1. 2.
Here, pigeons = the people, pigeonholes = the days of the week.
There are only 7 days in a week and 10 people, therefore at least two of them were born on the same day of the week.

2. By pigeonhole, there is a person who got at least $\lceil 202/3 \rceil = 68$ votes. So someone could win with a 67-67-68 split.

3. There are 10 digits, so there are $10^3 = 1000$ possibilities for the last three digits of a positive integer. If you write down 1001 numbers (the pigeon), then by the pigeonhole principle must have at least 2 numbers with the same last three digits.

4. There are 6 pigeonholes $H_1 = \{0, 10\}, H_2 = \{1, 9\}, H_3 = \{2, 8\}, H_4 = \{3, 7\}, H_5 = \{4, 6\}, H_6 = \{5\}$, and there are 7 pigeons (the elements of S). By pigeonhole principle, at least one element of S will be either H_1 to H_5 which sum to 10.

5. There are 6 pigeonholes $H_1 = \{1, 11\}, H_2 = \{2, 10\}, H_3 = \{3, 9\}, H_4 = \{4, 8\}, H_5 = \{5, 8\}, H_6 = \{6\}$, and there are 7 pigeons (the elements of S). BY pigeonhole principle, at least one element of S will be either H_1 to H_5 which sum to 12.

6. 27.
   Pigeonhole = 26 letters in the alphabet.
   Pigeon = first letter in the last name of the students.

7. A box has many red balls and blue balls. You draw balls. What is the smallest number of balls drawn that will guarantee
   - there are 2 of the same colour? 3
   - there are 3 of the same colour? 5
   - there are 11 of the same colour? 21

8. Each person knows either $\{1, 2, ..., 9\}$ other people or $\{0, 1, ..., 8\}$ other people.
   - We cannot have the case that one person knows nobody AND another person person knows everybody.

   In either case, we have 9 possibilities (pigeonholes) and 10 people (pigeon). By pigeonhole principle two of them will know the same number of people.

9. Solve $\lceil \frac{x}{131} \rceil = 6$ gives $x = 131$. Or, suppose a group of 26 people have all their first name initial different (i.e., a,b,c,d,...,z). When we have 5 such groups, each of the latter in the alphabet (i.e., a,b,c,d,...,z), there are 5 people in the room share such letter. Now $5 * 26 = 130$. To guarantee at least 6 share the same first name initial, we need 1 more, i.e., 131.

10. False. One hole could have all $n + 1$ pigeons.
    Note that Pigeonhole Principle is only "a guarantee on what can happen in the worst case scenario". It is not "the worst case scenario must occur".

11. False.
    There exists one box with at least that many, but it could contain more.

3 Discrete probability and statistics

3.1 Probability

1. Roll two six-sided fair dice. What is the probability that the sum of the numbers on the dice is 10? Find this probability without using generating function.

2. Roll two six-sided fair dice. What is the probability that the sum of the numbers on the dice is 10? Find this probability using generating function.

3. Five coins are randomly selected from 1000 coins among which 20 are fake. Find the probability of obtaining no fake coins.

4. Five coins are randomly selected from 1000 coins among which 20 are fake. Find the probability of obtaining at least one fake coin.

5. Randomly picking a positive integer between 1 and 100, what is the probability that a the integer is divisible by either 2 or 5?

6. Rolling a fair six-sided dice, what is the probability that the die comes up six, given it is even?

7. Rolling a fair six-sided dice six times. What is the probability that the die never comes up an even number ?

8. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding 52.

9. Find the probability of rolling at least one six when a fair six-sided die is rolled four times.

10. Suppose that a six-sided die is biased so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

11. What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Hint: use the definition of conditional probability).

12. Consider a family with two children, that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent? (Hint: use the definition of independent probability).
13. A coin is biased so that the probability of heads is $2/3$. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?
14. Given a set of three numbers $A = \{1, 2, 3\}$.
    - How many ways can you permute the three numbers of A?
    - List all the possible permutations the three numbers of A.
    - If we randomly permute the the three numbers of A, what is the probability that 1 precedes 3?
15. Given a set of four numbers $A = \{1, 2, 3, 4\}$. If we randomly permute the the four numbers of A, what is the probability that 1 precedes 4?
    - Solve this by brute force: list all the possible permutations.
    - Solve this analytically
16. Given a set of 100 numbers $A = \{1, 2, ..., 100\}$. If we randomly permute the the 100 numbers of A, what is the probability that 1 precedes 100? Explain how you get the answer.
17. Given a set of four numbers $A = \{1, 2, 3, 4\}$. If we randomly permute the the four numbers of A, what is the probability that 1 precedes 4 and 4 precedes 2?
    - Solve this by brute force: list all the possible permutations.
    - Solve this analytically
18. Given a set of five numbers $A = \{1, 2, 3, 4, 5\}$. If we randomly permute the the five numbers of A, what is the probability that 1 precedes 4 and 4 precedes 2?
19. Given a set of four numbers $A = \{1, 2, 3, 4\}$. If we randomly permute the the four numbers of A, what is the probability that 1 precedes 4 and 3 precedes 2?
    - Solve this by brute force: list all the possible permutations.
    - Solve this analytically

3.2 Random variable: probability, expectation and variance

1. Let X be the number that comes up when a fair six-sided die is rolled. What is the expected value of X?
2. A fair coin (with head $H = 1$ and tail $T = 0$) is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?
3. Two fair six-sided dice is rolled.
    - List the sample space (all the possible outcome) of the sum of the numbers.
    - What is the expected value of the sum of the numbers?
4. A random variable X has the probability distribution

x	0	1	2	3	4
$\mathbb{P}(X = x)$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{1}{24}$

where a is a constant.

    - find a
    - find $\mathbb{P}(X > 3)$
    - find $\mathbb{P}(X < 4)$
    - find $\mathbb{E}[X]$
    - find $\mathbb{E}[X^2]$
    - find $\mathbb{V}[X]$

5. A random variable X has the probability distribution

x	0	1	2	3
$\mathbb{P}(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

   - find $\mathbb{P}(1 < X \leq 3)$
   - find $\mathbb{E}[X]$
   - find $\mathbb{E}[2X-3]$
   - find $\mathbb{V}[X]$
   - find $\mathbb{V}[2X-3]$

6. A random variable X has the probability distribution $\mathbb{P}(X=x) = \begin{cases} ax^2 & x = 3,4,5 \\ 0 & \text{otherwise} \end{cases}$
   - find a
   - find $\mathbb{E}[5X-4]$
   - find $\mathbb{V}[5X-4]$

7. A discrete random variable X has mean 7 and variance 11.
   - find $\mathbb{E}[X^2]$
   - find $\mathbb{E}[2X-4]$
   - find $\mathbb{V}[2X-4]$

8. A random variable X has discrete uniform distribution, $X \sim U(a,b)$.
   Now let $a=1, b=10$, i.e., $\mathbb{P}(X=x) = \begin{cases} 1/10 & x = 1,2,3,...,10 \\ 0 & \text{otherwise} \end{cases}$
   - find $\mathbb{E}[X^2]$ by definition $\mathbb{E}[X^2] = \sum x^2 p(x)$. (Hint: $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$)
   - given that for $X \sim U(a,b)$, we have $\mathbb{E}[X] = \dfrac{a+b}{2}$ and $\mathbb{V}[X] = \dfrac{(b-a+1)^2-1}{12}$. Find $\mathbb{E}[X^2]$ by the formula of $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
   - find $\mathbb{P}(X+2 < 3X-4 < X+7)$
   - given $\mathbb{E}[kX+5] = 6.1$, find $\mathbb{V}(kX+5)$

9. A random variable X has continuous distribution, $\mathbb{P}(X=x) = \begin{cases} \dfrac{2}{75}x & 0 \leq x \leq 5 \\ \dfrac{2}{15} & 5 < x \leq 10 \\ 0 & \text{otherwise} \end{cases}$
   - sketch $\mathbb{P}(X=x)$ for all x
   - find $\mathbb{P}(X > 4)$
   - find $\mathbb{P}(3 \leq X \leq 4)$

10. A random variable X has continuous distribution, $\mathbb{P}(X=x) = \begin{cases} kx & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$
    - Show that $k = \dfrac{1}{72}$
    - find $\mathbb{P}(X > 5)$
    - show that $\mathbb{E}[X] = \mathbb{V}[X]$
    - sketch $\mathbb{P}(X=x)$ for all x

3.3 Two random variables

1. Consider the following table

	$X=1$	$X=2$	$X=3$
$Y=0$	0.1	0.1	0.2
$Y=1$	0.2	a	0.1

   (a) What is the sample space here?
   (b) What is $\mathbb{P}(X=1,Y=1)$ here?
   (c) What is $\mathbb{P}(X=3,Y=3)$ here?
   (d) Find the value of a
   (e) What is $\mathbb{P}(X=3)$ here?
   (f) What is $\mathbb{P}(Y=1)$ here?
   (g) What is all the marginal probability here?
   (h) What is $\mathbb{E}[X]$ here?
   (i) What is $\mathbb{E}[Y]$ here?
   (j) What is $\mathbb{E}[2X]$ here?
   (k) What is $\mathbb{E}[-3Y]$ here?
   (l) What is $\mathbb{E}[X^2]$ here?
   (m) What is $\mathbb{E}[Y^2]$ here?
   (n) What is $\mathbb{V}[X]$ here?
   (o) What is $\mathbb{V}[Y]$ here?
   (p) What is $\mathbb{E}[XY]$ here?
   (q) What is $\mathbb{E}[X+Y]$ here?
   (r) What is $\mathbb{E}[(X,Y)]$ here?
   (s) What is $\mathrm{cov}(X,Y)$ here?
   (t) What is $\mathrm{corr}(X,Y)$ here?
   (u) What is $\mathbb{P}(X=1|Y=0)$?
   (v) What is $\mathbb{P}(Y=0|X=1)$?
   (w) What is $\mathbb{E}[X|Y=1]$?
   (x) What is $\mathbb{E}[X|Y=0]$?
   (y) What is the probability distribution function for $\mathbb{E}[X|Y]$?
   (z) What is $\mathbb{V}[Y|X=1]$?

2. Consider the following table

	$X=a$	$X=1$
$Y=0$	b	0.1
$Y=1$	0.2	c

   Given $\mathbb{E}[Y]=0.5$ and $\mathbb{E}[XY]=0.8$, find the unknowns a,b,c.

3.4 Common distributions

1. Recall that the Binomial distribution is $p(m|\theta)=\binom{n}{m}\theta^m(1-\theta)^{n-m}$. Suppose a random variable X follows a binomial distribution with $n=20$ and $\theta=0.4$,
   - explain what is the physical meaning of $\theta=0.4$ here
   - find $\mathbb{P}(X=6)$
   - find $\mathbb{P}(X<3)$

2. A random variable X follows a binomial distribution $p(m|\theta)=\binom{n}{m}\theta^m(1-\theta)^{n-m}$. Given that $\mathbb{P}(X=2)=\mathbb{P}(X=3)$, show that $\mathbb{E}[X]=3-\theta$.

3.5 Solution to probability

1. There are six outcomes $X = \{1,2,3,4,5,6\}$ for one dice. For two dices, there are $6 \cdot 6 = 36$ outcomes, i.e., the sample space. Out of these 36 outcomes, the event is $\{(4,6),(5,5),(6,4)\}$. By the definition of classical probability, the desired probability is $\dfrac{|\{(4,6),(5,5),(6,4)\}|}{36} = \dfrac{3}{36} = \dfrac{1}{12}$.

2. The die is fair, so the generating function of rolling two die is $G(x) = \dfrac{(x+x^2+x^3+x^4+x^5+x^6)^2}{36}$.

   We are looking for $[x^{10}]G(x)$, or equivalently $[x^8]\dfrac{(1+x^2+x^3+x^4+x^5)^2}{36}$

$$
\begin{aligned}
(1+x^2+x^3+x^4+x^5)^2 = \Big(\frac{1-x^6}{1-x}\Big)^2 &= (1-x^6)^2\Big(\frac{1}{1-x}\Big)^2 \\
&= \Big(\sum_{k=0}^{2}\binom{2}{k}(-x^6)^k\Big)\Big(1+x+x^2+\ldots\Big)^2 \\
&= \Big(\sum_{k=0}^{2}\binom{2}{k}(-1)^k x^{6k}\Big)\Big(\sum_{r=0}^{\infty}\binom{r+2-1}{r}x^r\Big) \\
&= \Big(\sum_{k=0}^{2}\binom{2}{k}(-1)^k x^{6k}\Big)\Big(\sum_{r=0}^{\infty}\binom{r+1}{r}x^r\Big) \\
&= \Big(\sum_{k=0}^{2}\binom{2}{k}(-1)^k x^{6k}\Big)\Big(\sum_{r=0}^{\infty}\binom{r+1}{1}x^r\Big) \qquad \text{using } \binom{n}{k} = \binom{n}{n-k} \\
&= \sum_{k=0}^{2}\binom{2}{k}(-1)^k \sum_{r=0}^{\infty}\binom{r+1}{1}x^{r+6k}
\end{aligned}
$$

   Let $8 = r+6k$ so $r = 8-6k$

$$
[x^8]\sum_{k=0}^{2}\binom{2}{k}(-1)^k\sum_{r=0}^{\infty}\binom{r+1}{1}x^{r+6k} \overset{r=8-6k}{=} [x^8]\sum_{k=0}^{2}\binom{2}{k}(-1)^k\sum_{8-6k=0}^{\infty}\binom{8-6k+1}{1}x^8 = [x^8]\sum_{k=0}^{2}\binom{2}{k}(-1)^k\sum_{8-6k=0}^{\infty}\binom{9-6k}{1}x^8
$$

   $\dbinom{9-6k}{1}$ is nonzero for $k = 0,1$, hence

$$
\binom{2}{0}(-1)^0\binom{9-6(0)}{1} + \binom{2}{1}(-1)^1\binom{9-6(1)}{1} = 1\cdot 1\cdot 9 + 1\cdot(-1)\cdot 6 = 3
$$

   Scale back with $\dfrac{1}{36}$ gives $\dfrac{3}{36} = \dfrac{1}{12}$.

3. $\dfrac{\text{number of ways to pick 5 coins from 980 non-fake coins}}{\text{number of ways to pick 5 coins from 1000 coins}} = \dfrac{\binom{980}{5}}{\binom{1000}{5}} \approx 0.903$

4. $1 - \dfrac{\text{number of ways to pick 5 coins from 980 non-fake coins}}{\text{number of ways to pick 5 coins from 1000 coins}} = 1 - \dfrac{\binom{980}{5}}{\binom{1000}{5}} \approx 0.096$

5. Let E_1 be the event that the integer selected at random is divisible by 2. Let E_2 be the event that it is divisible by 5. Then $\mathbb{P}(E_1 \cup E_2)$ is what we want. By inclusion-exclusion principle,

$$
\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_1) - \mathbb{P}(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}
$$

6. $\dfrac{\frac{1}{6}}{\frac{3}{6}} = \dfrac{1}{3}$

7. $(\dfrac{3}{6})^6 = \dfrac{1}{2^6} = \dfrac{1}{64}$

8. $\dfrac{1}{\binom{52}{6}} = \dfrac{1}{20358520}$

9. $1 - \mathbb{P}\big(\text{no six for all the four roll}\big) = 1 - \Big(\dfrac{5}{6}\Big)^4 = \dfrac{671}{1296}$

10. We want to find the probability of the event $E = \{1, 3, 5\}$. From the question that 3 has double chance, so by $\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(4) = \mathbb{P}(5) = \frac{1}{7}$, $\mathbb{P}(3) = \frac{2}{7}$, therefore $\mathbb{P}(E) \;=\; \mathbb{P}(1) + \mathbb{P}(3) + \mathbb{P}(5) \;=\; \frac{1}{7} + \frac{2}{7} + \frac{1}{7} \;=\; \frac{4}{7}$.

11. Let E be the event that a family with two children has two boys, and let F be the event that a family with two children has at least one boy. It follows that $E = \{BB\}$, $F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$. Because the four possibilities are equally likely, it follows that $\mathbb{P}(F) = \frac{3}{4}$ and $\mathbb{P}(E \cap F) = \frac{1}{4}$. We conclude that
$$\mathbb{P}(E|F) \;=\; \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} \;=\; \frac{1/4}{3/4} = \frac{1}{3}.$$

12. Because $E = \{BB\}$, we have $\mathbb{P}(E) = 1/4$. In the last question we showed that $\mathbb{P}(F) = 3/4$ and that $\mathbb{P}(E \cap F) = 1/4$. But $\mathbb{P}(E)\mathbb{P}(F) = 1 \cdot 3 = 3$. Therefore $\mathbb{P}(E \cap F) \neq \mathbb{P}(E)\mathbb{P}(F)$, so the events E and F are not independent.

13. There are $2^7 = 128$ possible outcomes when a coin is flipped seven times. The number of ways four of the seven flips can be heads is $\binom{7}{4}$. Because the seven flips are independent, the probability of each of these outcomes (four heads and three tails) is $(\frac{2}{3})^4(\frac{1}{3})^3$. Consequently, the probability that exactly four heads appear is
$$\binom{7}{4}(\frac{2}{3})^4(\frac{1}{3})^3 \;=\; \frac{35 \cdot 16}{3^7} \;=\; \frac{560}{2187}.$$

14. $A = \{1, 2, 3\}$
    - The number of permutations of A is $3! = 6$.
    - All the permutations of A: $123, 132, 213, 231, 312, 321$
    - The permutations that 1 precedes 3 are 123, 132, 213, so the probability that 1 precedes 3 is $\frac{3}{6} = \frac{1}{2}$

15. $A = \{1, 2, 3, 4\}$
    - $\dfrac{\left|\left\{1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 3124, 3142, 2314, 3214\right\}\right|}{4!} = \frac{1}{2}$.

16. $\frac{1}{2}$. Regardless of the permutation, there are only two possibilities: $\{1$ precedes $100\}$ and $\{1$ NOT precedes $100\}$. Therefore the probability is $\frac{1}{2}$.

17. $A = \{1, 2, 3, 4\}$
    - $\dfrac{\left|\left\{1423, 1432, 1342, 3142\right\}\right|}{4!} = \frac{1}{6}$.
    - 1 precedes 4 and 4 precedes 2 is like putting 3 numbers in 4 boxes. There are 4 ways to put 3 numbers in 4 boxes: $1X42, 14X2, 142X$ and $X142$, or, using binomial coefficient, $\binom{4}{3}$. Hence there are $\dfrac{\binom{4}{3}}{4!} = \frac{1}{6}$ chance.

18. $\dfrac{\binom{5}{3}}{5!} = \frac{1}{6}$.
    There are $\binom{5}{3}$ ways to put 3 numbers in 5 boxes

19. Given a set of four numbers $A = \{1, 2, 3, 4\}$.
    - $\dfrac{\left|\left\{1432, 1342, 1324, 3214, 3124, 3142\right\}\right|}{4!} = \frac{1}{4}$.
    - $\dfrac{\binom{4}{2}}{4!}$. There are $\binom{4}{2}$ ways to put 2 pairs (2 boxes each) in 4 boxes.

3.6 Solution to random variable: probability, expectation and variance

1. $\mathbb{E}(X) = \sum xp(x) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + ... + \frac{1}{6}\cdot 6 = \frac{7}{2}$

2. $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$. Because the coin is fair and the flips are independent, the probability of each outcome is 1/8 and we have

$$\begin{aligned}
\mathbb{E}(X) &= \frac{X(HHH)+X(HHT)+X(HTH)+X(THH)+X(TTH)+X(THT)+X(HTT)+X(TTT)}{8} \\
&= \frac{3+2+2+2+1+1+1+0}{8} \\
&= \frac{3}{2}
\end{aligned}$$

3. Two fair six-sided dice is rolled.

   - The sample space is $\{2,3,4,5,6,7,8,9,10,11,12\}$
   - The probability of each element in the sample space

$$\begin{aligned}
\mathbb{P}(2) &= \frac{1}{36} & \mathbb{P}(3) &= \frac{2}{36} & \mathbb{P}(4) &= \frac{3}{36} & \mathbb{P}(5) &= \frac{4}{36} \\
\mathbb{P}(6) &= \frac{5}{36} & \mathbb{P}(7) &= \frac{6}{36} & \mathbb{P}(8) &= \frac{5}{36} & \mathbb{P}(9) &= \frac{4}{36} \\
\mathbb{P}(10) &= \frac{3}{36} & \mathbb{P}(11) &= \frac{2}{36} & \mathbb{P}(12) &= \frac{1}{36}
\end{aligned}$$

   - The expected value of the sum of the numbers

$$\mathbb{E}(X) = \sum x\mathbb{P}(x) = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + ... + 12\cdot\frac{1}{36} = 7$$

4. A random variable X has the probability distribution

x	0	1	2	3	4
$\mathbb{P}(X=x)$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{1}{24}$

where a is a constant.

   - find a

$$\begin{aligned}
& \mathbb{P}(\Omega) = 1 \\
\iff\quad & \sum_x \mathbb{P}(X=x) = 1 \\
\iff\quad & \mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3)+\mathbb{P}(X=4) = 1 \\
\iff\quad & \frac{3}{8}+\frac{1}{3}+\frac{1}{4}+a+\frac{1}{24} = 1 \\
\iff\quad & a = 0
\end{aligned}$$

   - find $\mathbb{P}(X>3)$

$$\mathbb{P}(X>3) = \mathbb{P}(X=4) = \frac{1}{24}$$

   - find $\mathbb{P}(X<4)$

$$\mathbb{P}(X<4) = \mathbb{P}(X=0,1,2,3) = \frac{3}{8}+\frac{1}{3}+\frac{1}{4}+0 = \frac{23}{24}$$

   or

$$\mathbb{P}(X<4) = 1-\mathbb{P}(X=4) = 1-\frac{1}{24} = \frac{23}{24}$$

   - find $\mathbb{E}[X]$

$$\mathbb{E}[X] = 0\cdot\frac{3}{8} + 1\cdot\frac{1}{3} + 2\cdot\frac{1}{4} + 3\cdot 0 + 4\cdot\frac{1}{24} = 1$$

- find $\mathbb{E}[X^2]$
$$\mathbb{E}[X^2] = 0^2 \cdot \frac{3}{8} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot 0 + 4^2 \cdot \frac{1}{24} = 2$$
- find $\mathbb{V}[X]$
$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2 - 1^2 = 1$$

5. A random variable X has the probability distribution

x	0	1	2	3
$\mathbb{P}(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

- find $\mathbb{P}(1 < X \leq 3)$
$$\mathbb{P}(1 < X \leq 3) = \mathbb{P}(X=2 \text{ or } X=3) = \mathbb{P}(X=2) + \mathbb{P}(X=3) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}.$$
(this test your understanding of probability in terms of random variable)
- find $\mathbb{E}[X]$
$$\mathbb{E}[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} = \frac{7}{8}$$
(this test whether you remember the definition of expectation)
- find $\mathbb{E}[2X-3]$
$$\mathbb{E}[2X-3] = 2\mathbb{E}[X] - 3 = \frac{14}{8} - 3 = \frac{14-24}{8} = \frac{-10}{8} = \frac{-5}{4}$$
(this test whether you remember that expectation is linear: $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$)
- find $\mathbb{V}[X]$
Method 1 (by definition)
$$\mathbb{V}[X] = \mathbb{E}\Big[(X - \mathbb{E}[X])^2\Big] = (0-\frac{7}{8})^2 \cdot \frac{1}{2} + (1-\frac{7}{8})^2 \cdot \frac{1}{4} + (2-\frac{7}{8})^2 \cdot \frac{1}{8} + (3-\frac{7}{8})^2 \cdot \frac{1}{8} = 1.109375$$
Method 2 (by formula)
$$\mathbb{V}[X] \;=\; \mathbb{E}[X^2] - {\color{red}(\mathbb{E}[X])^2} \;=\; \underbrace{(0)^2\frac{1}{2} + (1)^2\frac{1}{4} + (2)^2\frac{1}{8} + (3)^2\frac{1}{8}}_{=1.875} - {\color{red}\left(\frac{7}{8}\right)^2} \;=\; 1.109375.$$
- find $\mathbb{V}[2X-3]$
$$\mathbb{V}[2X-3] = 2^2\mathbb{V}[X] = 4\mathbb{V}[X] = 4(1.109375) = 4.4375$$
(this test whether you remember that $\mathbb{V}[aX+b] = a^2\mathbb{E}[X]$)

6. A random variable X has the probability distribution $\mathbb{P}(X=x) = \begin{cases} ax^2 & x = 3,4,5 \\ 0 & \text{otherwise} \end{cases}$

- find a
$$\begin{aligned} \text{probability of sample space is 1} \iff \mathbb{P}(\Omega) = 1 \iff \sum_x \mathbb{P}(X=x) = 1 &\iff \mathbb{P}(X=3) + \mathbb{P}(X=4) + \mathbb{P}(X=5) = 1 \\ &\iff a3^2 + a4^2 + a5^2 = 1 \\ &\iff a = 1/50 \end{aligned}$$
- find $\mathbb{E}[5X-4]$
$$\mathbb{E}[5X-4] = 5\mathbb{E}[X] - 4 = 5\underbrace{\Big(3 \cdot 9/50 + 4 \cdot 16/50 + 5 \cdot 25/50\Big)}_{=\mathbb{E}[X]=4.32} - 4 = 17.6$$

- find $\mathbb{V}[5X-4]$
  Method 1 (By definition)

$$
\begin{aligned}
\mathbb{V}[5X-4] \;=\; 5^2\mathbb{V}[X] \;&=\; 25\mathbb{E}\Big[(X-\mathbb{E}[X])^2\Big] \\
&=\; 25\left((3-4.32)^2\cdot\frac{3^2}{50}+(4-4.32)^2\cdot\frac{4^2}{50}+(5-4.32)^2\cdot\frac{5^2}{50}\right) \\
&=\; 25\left(\underbrace{(3-4.32)^2\cdot\frac{9}{50}+(4-4.32)^2\cdot\frac{16}{50}+(5-4.32)^2\cdot\frac{25}{50}}_{=0.5776}\right) \\
&=\; 14.44
\end{aligned}
$$

Place easy to have careless mistake: note that $\mathbb{P}(X=x)$ is ax^2 not a, so $\mathbb{P}(X=3)$ is $a3^2=9a$. It is easy to forget that there is a x^2 in $\mathbb{P}(X=x)$

Method 2 (By formula)

$$
\begin{aligned}
\mathbb{V}[5X-4] \;=\; 5^2\mathbb{V}[X] \;=\; 25\Big(\mathbb{E}[X^2]-(\mathbb{E}[X])^2\Big) \;&=\; 25\Big(3^2\cdot 9/50+4^2\cdot 16/50+5^2\cdot 25/50-4.32^2\Big) \\
&=\; 25\Big(19.24-18.6624\Big) \\
&=\; 14.44
\end{aligned}
$$

7. A discrete random variable X has mean 7 and variance 11.

   - find $\mathbb{E}[X^2]$
   $$\mathbb{E}[X^2] \;=\; \mathbb{V}[X]+(\mathbb{E}[X])^2 \;=\; 11+7^2 \;=\; 60.$$
   - find $\mathbb{E}[2X-4]$
   $$\mathbb{E}[2X-4]=2\mathbb{E}[X]-4=2\cdot 7-4=10.$$
   - find $\mathbb{V}[2X-4]$
   $$\mathbb{V}[2X-4]=2^2\mathbb{V}[X]=4\cdot 11=44.$$

8. A random variable X has discrete uniform distribution, $X\sim U(a,b)$.
   Now let $a=1, b=10$, i.e., $\mathbb{P}(X=x)=\begin{cases}1/10 & x=1,2,3,...,10\\ 0 & \text{otherwise}\end{cases}$

   - find $\mathbb{E}[X^2]$ by definition $\mathbb{E}[X^2]=\sum x^2p(x)$. (Hint: $1^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$)
   $$\mathbb{E}[X^2] \;=\; 1^2\cdot\frac{1}{10}+2^2\cdot\frac{1}{10}+...+10^2\cdot\frac{1}{10} \;=\; \frac{1^2+...+10^2}{10} \;=\; \frac{10(11)(21)}{6(10)} \;=\; 38.5$$
   - given that for $X\sim U(a,b)$, we have $\mathbb{E}[X]=\dfrac{a+b}{2}$ and $\mathbb{V}[X]=\dfrac{(b-a+1)^2-1}{12}$. Find $\mathbb{E}[X^2]$ by the formula of $\mathbb{V}[X]=\mathbb{E}[X^2]-(\mathbb{E}[X])^2$
   $$\mathbb{E}[X^2] \;=\; \mathbb{V}[X]+(\mathbb{E}[X])^2 \;=\; \frac{(10-1+1)^2-1}{12}+\left(\frac{1+10}{2}\right)^2 \;=\; \frac{99}{12}+\frac{121}{4} \;=\; 38.5$$
   - find $\mathbb{P}(X+2<3X-4<X+7)$
   $$\mathbb{P}(X+2<3X-4<X+7) \;=\; \mathbb{P}(2<2X-4<7) \;=\; \mathbb{P}(6<2X<11) \;=\; \mathbb{P}(3<X<11/2) \;=\; \mathbb{P}(X=4,5) \;=\; 1/5$$
   - given $\mathbb{E}[kX+5]=6.1$, find $\mathbb{V}[kX+5]$
   $$\mathbb{V}[kX+5] \;=\; k^2\mathbb{V}[X] \;=\; k^2\frac{99}{12} \;=\; \frac{33}{4}k^2.$$
   We need to find k by $\mathbb{E}[kX+5]=6.1$.
   $$\mathbb{E}[kX+5]=6.1 \iff k\mathbb{E}[X]+5=6.1 \iff k\mathbb{E}[X]=1.1 \iff k\frac{11}{2}=1.1 \iff k=0.2$$
   So $\mathbb{V}[kX+5]=\dfrac{33}{4}k^2=\dfrac{33}{4}0.2^2=0.33$

9. A random variable X has continuous distribution, $\mathbb{P}(X = x) = \begin{cases} \frac{2}{75}x & 0 \leq x \leq 5 \\ \frac{2}{15} & 5 < x \leq 10 \\ 0 & \text{otherwise} \end{cases}$

   - sketch $\mathbb{P}(X = x)$ for all x
   - find $\mathbb{P}(X > 4)$

   $$\mathbb{P}(X > 4) = 1 - \mathbb{P}(\text{NOT}\{X > 4\}) = 1 - \mathbb{P}(X \leq 4) = 1 - \int_0^4 p(x)dx = 1 - \int_0^4 \frac{2}{75}xdx = 1 - \frac{1}{75}x^2\Big|_0^4 = 1 - \frac{16}{79} = \frac{59}{79}$$

   Common mistake:

   $$\text{WRONG : } \mathbb{P}(X > 4) = \mathbb{P}(X \geq 5) = \int_5^{10} p(x)dx = \ldots$$

   Here X is a continuous variable, $X > 4$ does not means $X = 5, 6, \ldots$, here $X > 4$ means X is any real number beyond 4

   - find $\mathbb{P}(3 \leq X \leq 4)$

   $$\mathbb{P}(3 \leq X \leq 4) = \mathbb{P}(X \leq 4) - \mathbb{P}(X \leq 3) = \int_0^4 \frac{2}{75}xdx - \int_0^3 \frac{2}{75}xdx = \frac{1}{75}x^2\Big|_0^4 - \frac{1}{75}x^2\Big|_0^3 = \frac{16}{75} - \frac{9}{75} = \frac{7}{75}$$

10. A random variable X has continuous distribution, $\mathbb{P}(X = x) = \begin{cases} kx & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$

   - Show that $k = \frac{1}{72}$

   $$\mathbb{P}(\Omega) = 1 \iff \int_X p(x)dx = 1 \iff \int_0^{12} kxdx = 1 \iff \frac{k}{2}x^2\Big|_0^{12} = 1 \iff k = \frac{1}{72}$$

   - find $\mathbb{P}(X > 5)$

   $$\mathbb{P}(X > 5) = 1 - \mathbb{P}(X \leq 5) = 1 - \int_0^5 \frac{1}{72}xdx = 1 - \frac{1}{144}x^2\Big|_0^5 = \frac{119}{144}$$

   - show that $\mathbb{E}[X] = \mathbb{V}[X]$

   $$\mathbb{E}[X] = \int_0^{12} xp(x)dx = \int_0^{12} \frac{1}{72}x^2dx = \frac{1}{72}\frac{1}{3}x^3\Big|_0^{12} = 8$$

   $$\mathbb{E}[X^2] = \int_0^{12} x^2p(x)dx = \int_0^{12} \frac{1}{72}x^3dx = \frac{1}{72}\frac{1}{4}x^4\Big|_0^{12} = 72$$

   $$\mathbb{V}[E] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 72 - 8^2 = 8 = \mathbb{E}[X]$$

   - sketch $\mathbb{P}(X = x)$ for all x

3.7 Solution to two random variables

1. Consider the following table

	$X = 1$	$X = 2$	$X = 3$
$Y = 0$	0.1	0.1	0.2
$Y = 1$	0.2	a	0.1

   (a) What is the sample space here?
   $(\mathcal{X}, \mathcal{Y}) = \Big\{(1,0), (1,1), (2,0), (2,1), (3,0), (3,1)\Big\}$

   (b) What is $\mathbb{P}(X = 1, Y = 1)$ here?
   0.2

   (c) What is $\mathbb{P}(X = 3, Y = 3)$ here?
   undefined or 0

   (d) Find the value of a
   By sum of all probability is 1, we have $a = 1 - 0.1 - 0.1 - 0.2 - 0.2 - 0.1 = 0.3$

(e) What is $\mathbb{P}(X = 3)$ here?
$0.2 + 0.1 = 0.3$

(f) What is $\mathbb{P}(Y = 1)$ here?
$0.2 + 0.3 + 0.1 = 0.6$

(g) What is all the marginal probability here?
On X, collapse along Y gives $\mathbb{P}(X = 1) = 0.3$, $\mathbb{P}(X = 2) = 0.4$ and $\mathbb{P}(X = 3) = 0.3$
On Y, collapse along X gives $\mathbb{P}(Y = 0) = 0.4$, $\mathbb{P}(Y = 1) = 0.6$

(h) What is $\mathbb{E}[X]$ here?
$\mathbb{E}[X] = 1(0.3) + 2(0.4) + 3(0.3) = 2$

(i) What is $\mathbb{E}[Y]$ here?
$\mathbb{E}[Y] = 0(0.4) + 1(0.6) = 0.6$

(j) What is $\mathbb{E}[2X]$ here?
$\mathbb{E}[2X] = 2\mathbb{E}[X] = 2(2) = 4$

(k) What is $\mathbb{E}[-3Y]$ here?
$\mathbb{E}[-3Y] = -3\mathbb{E}[Y] = -3(0.6) = -1.8$

(l) What is $\mathbb{E}[X^2]$ here?
$\mathbb{E}[X^2] = 1^2(0.3) + 2^2(0.4) + 3^2(0.3) = 4.6$

(m) What is $\mathbb{E}[Y^2]$ here?
$\mathbb{E}[Y^2] = 0^2(0.4) + 1^2(0.6) = 0.6$

(n) What is $\mathbb{V}[X]$ here?
$\mathbb{V}[X] = (1-2)^2(0.3) + (2-2)^2(0.4) + (3-2)^2(0.3) = 0.6$

Method 2 (formula) $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \overset{(h),(\ell)}{=} 4.6 - (2)^2 = 4.6 - 4 = 0.6$

(o) What is $\mathbb{V}[Y]$ here?
$\mathbb{V}[Y] = (0-0.6)^2(0.4) + (1-0.6)^2(0.6) = 0.24$

Method 2 (formula) $\mathbb{V}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \overset{(i),(m)}{=} 0.6 - (0.6)^2 = 0.6 - 0.36 = 0.24$

(p) What is $\mathbb{E}[XY]$ here?
$1(0)(0.1) + 2(0)(0.1) + 3(0)(0.2) + 1(1)(0.2) + 2(1)(0.3) + 3(1)(0.1) = 1.1$

(q) What is $\mathbb{E}[X+Y]$ here?
$(1+0)(0.1) + (2+0)(0.1) + (3+0)(0.2) + (1+1)(0.2) + (2+1)(0.3) + (3+1)(0.1) = 2.6$

(r) What is $\mathbb{E}[(X,Y)]$ here?
$(1,0)(0.1) + (2,0)(0.1) + (3,0)(0.2) + (1,1)(0.2) + (2,1)(0.3) + (3,1)(0.1) = (2, 0.6)$
You can also think of (X,Y) as a 2-dimensional vector $\begin{bmatrix} X \\ Y \end{bmatrix}$, so this question is the same as asking about "what is the expected value of the vector $\begin{bmatrix} X \\ Y \end{bmatrix}$?"

(s) What is $\mathrm{cov}(X,Y)$ here?
$\mathrm{cov}(X,Y) = \mathbb{E}\Big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\Big]$, so

	$X - \mathbb{E}[X] = -1$	$X - \mathbb{E}[X] = 0$	$X - \mathbb{E}[X] = 1$
$Y - \mathbb{E}[Y] = -0.6$	0.1	0.1	0.2
$Y - \mathbb{E}[Y] = 0.4$	0.2	0.3	0.1

$$\begin{aligned} \mathbb{E}\Big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\Big] &= (-1)(-0.6)(0.1) + (0)(-0.6)(0.1) + (1)(-0.6)(0.2) \\ &\quad +(-1)(0.4)(0.2) + (0)(0.4)(0.3) + (1)(0.4)(0.1) \\ &= -0.1 \end{aligned}$$

Method 2 (formula) $\mathbb{E}\Big[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\Big] = \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y] \overset{(h),(i),(p)}{=} 1.1-2(0.6) = -0.1$

(t) What is $\text{corr}(X,Y)$ here?

$$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\mathbb{V}[X]\mathbb{V}[Y]}} = \frac{-0.1}{\sqrt{0.6\cdot 0.24}} = -0.263$$

(u) What is $\mathbb{P}(X=1|Y=0)$?

$$\mathbb{P}(X=1|Y=0) = \frac{\mathbb{P}(X=1,Y=0)}{\mathbb{P}(Y=0)} = \frac{0.1}{0.4} = \frac{1}{4}$$

(v) What is $\mathbb{P}(Y=0|X=1)$?

$$\mathbb{P}(Y=0|X=1) = \frac{\mathbb{P}(Y=0,X=1)}{\mathbb{P}(X=1)} = \frac{0.1}{0.2+0.1} = \frac{1}{3}$$

(w) What is $\mathbb{E}[X|Y=1]$?

$$\begin{aligned}
\mathbb{E}[X|Y=1] &= 1\cdot\mathbb{P}(X=1|Y=1)+2\cdot\mathbb{P}(X=2|Y=1)+3\cdot\mathbb{P}(X=3|Y=1)\\
&= \frac{\mathbb{P}(X=1,Y=1)}{\mathbb{P}(Y=1)}+2\cdot\frac{\mathbb{P}(X=2,Y=1)}{\mathbb{P}(Y=1)}+3\cdot\frac{\mathbb{P}(X=3,Y=1)}{\mathbb{P}(Y=1)}\\
&= \frac{0.2}{0.6}+2\cdot\frac{0.3}{0.6}+3\cdot\frac{0.1}{0.6}\\
&= \frac{11}{6} = 1.83333
\end{aligned}$$

(x) What is $\mathbb{E}[X|Y=0]$?

$$\begin{aligned}
\mathbb{E}[X|Y=0] &= 1\cdot\mathbb{P}(X=1|Y=0)+2\cdot\mathbb{P}(X=2|Y=0)+3\cdot\mathbb{P}(X=3|Y=0)\\
&= \frac{\mathbb{P}(X=1,Y=0)}{\mathbb{P}(Y=0)}+2\cdot\frac{\mathbb{P}(X=2,Y=0)}{\mathbb{P}(Y=0)}+3\cdot\frac{\mathbb{P}(X=3,Y=0)}{\mathbb{P}(Y=0)}\\
&= \frac{0.1}{0.4}+2\cdot\frac{0.1}{0.4}+3\cdot\frac{0.2}{0.4}\\
&= 2.25
\end{aligned}$$

(y) What is the probability distribution function for $\mathbb{E}[X|Y]$?

$$\mathbb{E}[X|Y] = \begin{cases} 2.25 & \text{if } Y=0 \text{ with probability } 0.4\\ 1.8333 & \text{if } Y=1 \text{ with probability } 0.6\end{cases}$$

(z) What is $\mathbb{V}[Y|X=1]$?

$$\begin{aligned}
\mathbb{V}[Y|X=1] &= \Big(0-\mathbb{E}[Y|X=1]\Big)^2\cdot p(0|X=1)+\Big(1-\mathbb{E}[Y|X=1]\Big)^2\cdot p(1|X=1)\\
&= \Big(\mathbb{E}[Y|X=1]\Big)^2\cdot\frac{\mathbb{P}(Y=0,X=1)}{\mathbb{P}(X=1)}+\Big(1-\mathbb{E}[Y|X=1]\Big)^2\cdot\frac{\mathbb{P}(Y=1,X=1)}{\mathbb{P}(X=1)}\\
&= \Big(\mathbb{E}[Y|X=1]\Big)^2\cdot\frac{0.1}{0.3}+\Big(1-\mathbb{E}[Y|X=1]\Big)^2\cdot\frac{0.2}{0.3}\\
&= \frac{\Big(\mathbb{E}[Y|X=1]\Big)^2}{3}+\frac{2}{3}\Big(1-\mathbb{E}[Y|X=1]\Big)^2
\end{aligned}$$

To proceed we need to compute $\mathbb{E}[Y|X=1]$

$$\mathbb{E}[Y|X=1] = 0\cdot p(0|X=1)+1\cdot p(1|X=1) = p(1|X=1) = \frac{2}{3}$$

Lastly,

$$\mathbb{V}[Y|X=1] = \frac{\Big(\mathbb{E}[Y|X=1]\Big)^2}{3}+\frac{2}{3}\Big(1-\mathbb{E}[Y|X=1]\Big)^2 = \frac{4/9}{3}+\frac{2}{3}\cdot\frac{1}{9} = \frac{2}{9}$$

2. Consider the following table

	$X = a$	$X = 1$
$Y = 0$	b	0.1
$Y = 1$	0.2	c

Given $\mathbb{E}[Y] = 0.5$ and $\mathbb{E}[XY] = 0.8$, find the unknowns a, b, c.

$$\begin{array}{lcl} \mathbb{E}[Y] = 0.5 = 0 \cdot (b + 0.1) + 1 \cdot (0.2 + c) = 0.5 & \iff & c = 0.3 \\ b + c + 0.3 = 1 & \iff & b = 0.4 \\ \mathbb{E}[XY] = a \cdot 0 \cdot b + 1 \cdot 0 \cdot 0.1 + a \cdot 1 \cdot 0.2 + 1 \cdot 1 \cdot c = 0.8 & \iff & a = 2.5 \end{array}$$

3.8 Solution to common distributions

1. Recall that the Binomial distribution is $p(m|\theta) = \binom{n}{m}\theta^m(1-\theta)^{n-m}$. Suppose a random variable X follows a binomial distribution with $n = 20$ and $\theta = 0.4$,

   - explain what is the physical meaning of $\theta = 0.4$ here

     the chance of success of 1 trial is 40%

   - find $\mathbb{P}(X = 6)$

$$\mathbb{P}(X = 6) = p(6|0.4) = \binom{20}{6}0.4^6(1-0.4)^{20-6} = 0.12441169921$$

   - find $\mathbb{P}(X < 3)$

$$\begin{aligned} \mathbb{P}(X < 3) = \mathbb{P}(X \leq 2) &= \mathbb{P}(X = 0 \text{ or } X = 1 \text{ or } X = 2) \\ &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) \\ &= p(0|0.4) + p(1|0.4) + p(2|0.4) \\ &= \binom{20}{0}0.4^0(1-0.4)^{20} + \binom{20}{1}0.4^1(1-0.4)^{20-1} + \binom{20}{2}0.4^2(1-0.4)^{20-2} \\ &= 0.00361147205 \end{aligned}$$

   (A common careless mistake: forgot the case $X = 0$.)

2. A random variable X follows a binomial distribution $p(m|\theta) = \binom{n}{m}\theta^m(1-\theta)^{n-m}$. Given that $\mathbb{P}(X = 2) = \mathbb{P}(X = 3)$, show that $\mathbb{E}[X] = 3 - \theta$. (Hint: the expected value of a binomial random variable is $n\theta$)

$$\begin{aligned} \mathbb{P}(X = 2) = \mathbb{P}(X = 3) &\iff \binom{n}{2}\theta^2(1-\theta)^{n-2} = \binom{n}{3}\theta^3(1-\theta)^{n-3} \\ &\iff \binom{n}{2} = \binom{n}{3}\theta(1-\theta)^{-1} \\ &\iff \frac{n!}{(n-2)!2!} = \frac{n!}{(n-3)!3!}\frac{\theta}{1-\theta} \\ &\iff \frac{1}{n-2} = \frac{1}{3}\frac{\theta}{1-\theta} \\ &\iff 3 - 3\theta = (n-2)\theta \\ &\iff 3 - p = n\theta = \mathbb{E}[X] \text{ by hint.} \end{aligned}$$

4 Continuous probability and statistics

4.1 Normal random variable

1. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the mean of $2X_1 - X_2$?
2. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the variance of $2X_1 - X_2$ if $\text{cov}(X_1, X_2) = -3$?
3. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the variance of $2X_1 - X_2$ if X_1 and X_2 are independent?

4. (**Hard**) Given $X_1 \sim \mathcal{N}(0, 2^2)$, $X_2 \sim \mathcal{N}(3, 3^2)$ and $X_3 \sim \mathcal{N}(-1, 1)$ are independent random variables. Let $Y = (X_1 + X_2 + X_3)^2$. Find $\mathbb{E}[Y]$

   (A side note, if X is a normal random variable, then X^2 is not a normal random variable but a chi-squared random variable. I.e., $X^2 \sim \chi^2$. We do not talk about chi-square statistics in this course.)

4.2 Confidence interval and hypothesis testing

1. The numbers of passengers on a bus U1C during 12:00-13:00 on 12 randomly chosen days are

$$47,\ 66,\ 55,\ 53,\ 49,\ 65,\ 48,\ 44,\ 50,\ 61,\ 60,\ 55$$

   - Estimate the mean daily number of passengers of bus U1C during 12:00-13:00.
   - Estimate the (unbiased) standard deviation of the daily number of passengers of bus U1C during 12:00-13:00.
   - Give a 95 percent confidence interval for the mean number of daily passengers of bus U1C during 12:00-13:00. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)
   - Explain the meaning of such interval.

2. The numbers of passengers on a bus U9 during 12:00-13:00 on 7 randomly chosen days are

$$37,\ 42,\ 51,\ 39,\ 44,\ 48,\ 29$$

   - Estimate the mean daily number of passengers of bus U9 during 12:00-13:00.
   - Estimate the (unbiased) standard deviation of the daily number of passengers of bus U9 during 12:00-13:00.
   - Give a 95 percent confidence interval for the mean number of daily passengers of bus U9 during 12:00-13:00. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)

3. Following Q1 and Q2, find the approximate 95% confidence interval for the difference of mean between bus U1C and U9. Explain the meaning of the confidence interval. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)

4. Following Q1-Q3, in a city council meeting, Peter wants to argue that that U1C is more crowded than U9 and therefore there should be more resource to U1C. He want to prove his argument that U1C and U9 have different degree of crowdedness.

   - Write down the null hypothesis and the alternative hypothesis
   - Find the p-value: write down the integral of finding the p-value. You do not need to solve the integral.

5. The table below shows the flat price (in 1000£) of a 2-bed flat.

City	n	the flat price (in 1000£)
London	8	$595, 395, 218, 370, 250, 435, 351, 124$
Southampton	5	$292, 425, 125, 140, 230$

   (a) Find the 95% confidence interval of mean flat price in London

   (b) Find the 95% confidence interval of mean flat price in Southampton

   (c) Find the approximate 95% confidence interval of difference in mean flat price between London and Southampton

   (d) You make a hypothesis that the flat price in London and the the flat price in Southampton is different. Write down the null hypothesis and alternative hypothesis, and find the p-value.

4.3 Solution to normal random variable

1. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the mean of $2X_1 - X_2$?
   Let $Y = 2X_1 - X_2$. Then

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[2X_1 - X_2] \\
&= \mathbb{E}[2X_1 + (-1)X_2] \\
&= 2\mathbb{E}[X_1] + (-1)\mathbb{E}[X_2] && \text{by expectation is linear } \mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y] \\
&= 2 \cdot 0 + (-1) \cdot 3 && \because X_1 \sim \mathcal{N}(0, 2^2), X_2 \sim \mathcal{N}(3, 3^2) \\
&= -3
\end{aligned}$$

   The mean of $2X_1 - X_2$ is -3

2. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the variance of $2X_1 - X_2$ if $\text{cov}(X_1, X_2) = -3$?
   Let $Y = 2X_1 - X_2$. Then

$$\begin{array}{rcll} \mathbb{V}[Y] &=& \mathbb{V}[2X_1 - X_2] & \\ &=& 2^2\mathbb{V}[X_1] - 2(2)(1)\text{cov}(X_1, X_2) + 1^2\mathbb{V}[X_2] & \mathbb{V}[aX - bY] = a^2\mathbb{V}[X] - 2ab\text{cov}(X, Y) + b^2\mathbb{V}[Y] \\ &=& 4 \cdot 2^2 - 4(-3) + 3^2 & \because X_1 \sim \mathcal{N}(0, 2^2), \text{cov}(X_1, X_2) = -3, X_2 \sim \mathcal{N}(3, 3^2) \\ &=& 37 & \end{array}$$

   The variance of $2X_1 - X_2$ is 37 if $\text{cov}(X_1, X_2) = -3$.

3. If $X_1 \sim \mathcal{N}(0, 2^2)$ and $X_2 \sim \mathcal{N}(3, 3^2)$, what is the variance of $2X_1 - X_2$ if X_1 and X_2 are independent?
   Let $Y = 2X_1 - X_2$. Then

$$\begin{array}{rcll} \mathbb{V}[Y] &=& \mathbb{V}[2X_1 - X_2] & \\ &=& 2^2\mathbb{V}[X_1] - 2(2)(1)\text{cov}(X_1, X_2) + 1^2\mathbb{V}[X_2] & \mathbb{V}[aX - bY] = a^2\mathbb{V}[X] - 2ab\text{cov}(X, Y) + b^2\mathbb{V}[Y] \\ &=& 4 \cdot 2^2 + 3^2 & \because X_1 \sim \mathcal{N}(0, 2^2), \text{cov}(X_1, X_2) = 0, X_2 \sim \mathcal{N}(3, 3^2) \\ &=& 25 & \end{array}$$

   The variance of $2X_1 - X_2$ is 25 if X_1 and X_2 are independent.

4. (**Hard**) Given $X_1 \sim \mathcal{N}(0, 2^2)$, $X_2 \sim \mathcal{N}(3, 3^2)$ and $X_3 \sim \mathcal{N}(-1, 1)$ are independent random variables. Let $Y = (X_1 + X_2 + X_3)^2$. Find $\mathbb{E}[Y]$.

$$\begin{array}{rcll} \mathbb{E}[Y] &=& \mathbb{E}\Big[(X_1 + X_2 + X_3)^2\Big] & \\ &=& \mathbb{E}\Big[X_1^2 + 2X_1X_2 + 2X_1X_3 + X_2^2 + 2X_2X_3 + X_3^2\Big] & \text{Trinomial expansion} \\ &=& \mathbb{E}[X_1^2] + 2\mathbb{E}[X_1X_2] + 2\mathbb{E}[X_1X_3] + \mathbb{E}[X_2^2] + 2\mathbb{E}[X_2X_3] + \mathbb{E}[X_3^2] & \text{expectation is linear} \\ &=& \mathbb{E}[X_1^2] + 2\mathbb{E}[X_1]\mathbb{E}[X_2] + 2\mathbb{E}[X_1]\mathbb{E}[X_3] + \mathbb{E}[X_2^2] + 2\mathbb{E}[X_2]\mathbb{E}[X_3] + \mathbb{E}[X_3^2] & X_1, X_2, X_3 \text{ are independent} \\ &=& \mathbb{E}[X_1^2] + 2(0)(3) + 2(0)(-1) + \mathbb{E}[X_2^2] + 2(3)(-1) + \mathbb{E}[X_3^2] & \\ &=& -6 + \mathbb{E}[X_1^2] + \mathbb{E}[X_2^2] + \mathbb{E}[X_3^2] & \end{array}$$

   Now tricky step: $\mathbb{V}[X] = \mathbb{E}[X^2] - \Big(\mathbb{E}[X]\Big)^2$ gives $\mathbb{E}[X^2] = \mathbb{V}[X] + \Big(\mathbb{E}[X]\Big)^2$, hence

$$\mathbb{E}[Y] \;=\; -6 + \mathbb{E}[X_1^2] + \mathbb{E}[X_2^2] + \mathbb{E}[X_3^2] \;=\; -6 + \Big(2^2 + 0^2\Big) + \Big(3^2 + 3^2\Big) + \Big(1^2 + (-1)^2\Big) \;=\; 22.$$

4.4 Solution to confidence interval and hypothesis testing

1. The numbers of passengers on a bus U1C during 12:00-13:00 on 12 randomly chosen days are

$$47,\ 66,\ 55,\ 53,\ 49,\ 65,\ 48,\ 44,\ 50,\ 61,\ 60,\ 55$$

   - Estimate the mean daily number of passengers of bus U1C during 12:00-13:00.

$$\text{sample mean} = \overline{x} = \frac{47 + 66 + 55 + 53 + 49 + 65 + 48 + 44 + 50 + 61 + 60 + 55}{12} = 54.42.$$

   - Estimate the (unbiased) standard deviation of the daily number of passengers of bus U1C during 12:00-13:00.
     The (unbiased) estimator of variance is

$$s^2 \;=\; \frac{1}{12-1}\sum_{i=1}^{12}(x_i - \overline{x})^2 \;=\; \frac{(47 - 54.42)^2 + (66 - 54.42)^2 + \cdots + (55 - 54.42)^2}{11} \;=\; 52.446 = 7.242^2.$$

     The (unbiased) standard deviation is 7.242.

   - Give a 95 percent confidence interval for the mean number of daily passengers of bus U1C during 12:00-13:00. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)

$$\Big[\overline{x} - 1.96\frac{s}{\sqrt{n}},\ \ \overline{x} + 1.96\frac{s}{\sqrt{n}}\Big] \;=\; \Big[54.42 - 1.96\frac{7.242}{\sqrt{12}},\ \ 54.42 + 1.96\frac{7.242}{\sqrt{12}}\Big] \;=\; \Big[50.3225,\ \ 58.5175\Big]$$

   - Explain the meaning of such interval.

     Repeating the same experiment (checking how many people in the bus U1C in 12:00-13:00) 100 times,
     on average, for 95 out of the 100 experiments
     the average number of people in bus U1C in 12:00-13:00 is between 51 and 58

2. The numbers of passengers on a bus U9 during 12:00-13:00 on 7 randomly chosen days are

$$37,\ 42,\ 51,\ 39,\ 44,\ 48,\ 29$$

- Estimate the mean daily number of passengers of bus U9 during 12:00-13:00.

$$\overline{x}_{U9} = 41.43$$

- Estimate the (unbiased) standard deviation of the daily number of passengers of bus U9 during 12:00-13:00.

$$s_{U9}^2 = 7.323^2 = 53.619$$

- Give a 95 percent confidence interval for the mean number of daily passengers of bus U9 during 12:00-13:00. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)

$$\left[41.43 - 1.96\frac{7.323}{\sqrt{7}},\quad 41.43 + 1.96\frac{7.323}{\sqrt{7}},\right] \;=\; \left[36.005,\quad 46.855\right]$$

3. Following Q1 and Q2, find the approximate 95% confidence interval for the difference of mean between bus U1C and U9. Explain the meaning of the confidence interval. (Hint: $z_{\alpha/2} = 1.96$ for $\alpha = 0.05$)

$$\begin{aligned}
T &= \left[\overline{x}_{U1C} - \overline{x}_{U9} + -1.96\sqrt{\frac{s_{U1C}^2}{n_{U1C}} + \frac{s_{U9}^2}{n_{U9}}},\quad \overline{x}_{U1C} - \overline{x}_{U9} + 1.96\sqrt{\frac{s_{U1C}^2}{n_{U1C}} + \frac{s_{U9}^2}{n_{U9}}}\right] \\
&= \left[54.42 - 41.43 - 1.96\sqrt{\frac{7.242^2}{12} + \frac{7.323^2}{7}},\quad 54.42 - 41.43 + 1.96\sqrt{\frac{7.242^2}{12} + \frac{7.323^2}{7}}\right] \\
&= \left[12.99 - 6.79,\quad 12.99 + 6.79\right] \\
&= \left[6.2,\quad 19.78\right]
\end{aligned}$$

Repeating the same experiment (checking how many people in the bus U1C and U9 in 12:00-13:00) 100 times,
on average, for 95 out of the 100 experiments
the average number of people in bus U1C in 12:00-13:00 is 7 to 19 more people than the bus U9

4. Following Q1-Q3, in a city council meeting, Peter wants to argue that that U1C is more crowded than U9 and therefore there should be more resource to U1C. He want to prove his argument that U1C and U9 have different degree of crowdedness.

- Write down the null hypothesis and the alternative hypothesis

$$H_0 \;:\; \mu_{U1C} = \mu_{U9} \qquad vs \qquad H_A \;:\; \mu_{U1C} \neq \mu_{U9}$$

- Find the p-value: write down the integral of finding the p-value. You do not need to solve the integral.

$$p = 2\mathbb{P}\Big(Z < -|z^*|\Big) \text{ where } z^* = \frac{\overline{x}_{U1C} - \overline{x}_{U9}}{\sqrt{\dfrac{s_{U1C}^2}{n_{U1C}} + \dfrac{s_{U9}^2}{n_{U9}}}}$$

Computing z^* gives $z^* = \dfrac{12.99}{3.4642} = 3.74969072165.$

Therefore, let $p(z)$ be the PDF of standard normal variable, we have

$$\begin{aligned}
p &= 2\mathbb{P}\Big(Z < -|3.74969072165|\Big) \\
&= 2\mathbb{P}\Big(Z < -3.74969072165\Big) \\
&= 2\int_{-\infty}^{-3.74969072165} p(z)dz \\
&= 2\int_{-\infty}^{-3.74969072165} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= \sqrt{\frac{2}{\pi}}\underbrace{\int_{-\infty}^{-3.74969072165} e^{-z^2/2}dz}_{=0.0002219} \\
&= 0.0001
\end{aligned}$$

The p-value is 0.0001, **very strong evidence against the null**.
I.e., by hypothesis testing, we have very strong evidence that "U1C and U9 during 12:00-13:00 is equally crowded on average" is false.

5. The table below shows the flat price (in 1000£) of a 2-bed flat.

City	n	the flat price (in 1000£)
London	8	$595, 395, 218, 370, 250, 435, 351, 124$
Southampton	5	$292, 425, 125, 140, 230$

(a) Find the 95% confidence interval of mean flat price in London

$$\begin{aligned}
\hat{\mu}_{\text{London}} = \overline{x}_{\text{London}} &= \frac{595+395+218+370+250+435+351+124}{8} = 342 \\
\hat{\sigma}^2_{\text{unbiased}} = s^2_{\text{London}} &= \frac{(595-342)^2+(395-342)^2+\cdots+(124-342)^2}{8-1} = 145.256^2 = 21099 \\
T_{0.05} &= \left[342 - 1.96\frac{145.526}{\sqrt{8}}, \quad 342 + 1.96\frac{145.526}{\sqrt{8}}\right] = \left[241.156, \quad 442.844\right]
\end{aligned}$$

(b) Find the 95% confidence interval of mean flat price in Southampton

$$\begin{aligned}
\hat{\mu}_{\text{Soton}} = \overline{x}_{\text{Soton}} &= \frac{292+425+125+140+230}{5} = 242.2 \\
\hat{\sigma}^2_{\text{unbiased}} = s^2_{\text{Soton}} &= \frac{(292-242.2)^2+(425-242.2)^2+\cdots+(230-242.2)^2}{8-1} = 122.7^2 = 15055 \\
T_{0.05} &= \left[242.2 - 1.96\frac{122.7}{\sqrt{5}}, \quad 242.2 + 1.96\frac{122.7}{\sqrt{5}}\right] = \left[134.64, \quad 349.75\right]
\end{aligned}$$

(c) Find the approximate 95% confidence interval of difference in mean flat price between London and Southampton

$$\left[(342-242.2) - 1.96\sqrt{\frac{145.256^2}{8}+\frac{122.7^2}{5}}, \quad (342-242.2) + 1.96\sqrt{\frac{145.256^2}{8}+\frac{122.7^2}{5}}\right] = \left[-47.506, \quad 247.106\right]$$

The confidence interval contains zero: that means we have the conclusion that "London and Southampton have the same average flat price".

This result contradicts to the common sense that "London is more expensive to live than Soton." This is probably due to the fact that we did not use **enough data points** to find the confidence interval (i.e., 8 and 5 points are too few and they do not relfect the real situation enough). The lesson here is that "we need more data".

(d) You make a hypothesis that the flat price in London and the the flat price in Southampton is different. Write down the null hypothesis and alternative hypothesis, and find the p-value.

$H_0 : \mu_{\text{Lodon}} = \mu_{\text{Soton}}$ and $H_A : \mu_{\text{Lodon}} \neq \mu_{\text{Soton}}$

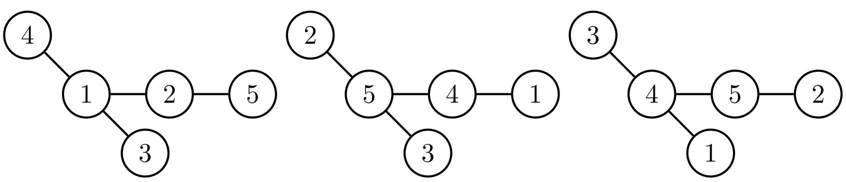
$$p = 2\mathbb{P}\left(Z < -\left|\frac{\overline{x}_{\text{London}} - \overline{x}_{\text{Soton}}}{\sqrt{\dfrac{s^2_{\text{London}}}{n_{\text{London}}} + \dfrac{s^2_{\text{Soton}}}{n_{\text{Soton}}}}}\right|\right) = 2\mathbb{P}(Z < -1.3279) = 2\underbrace{\int_{-\infty}^{-1.3279}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz}_{0.092} = 0.184$$

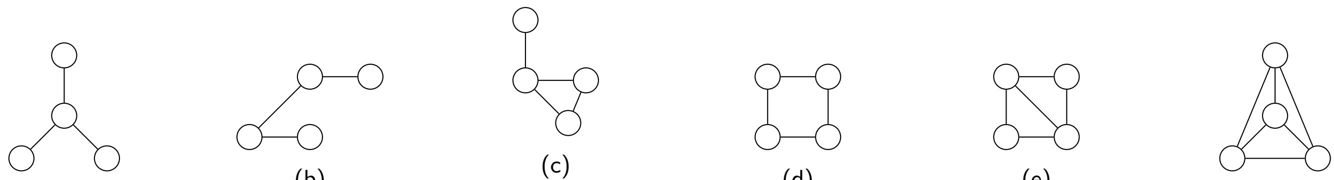
The p-value is 0.184, we have insufficient evidence to against the null hypothesis.
I.e., we have insufficient evidence to against "London and Soton have different flat price".
Again, we can blame that "we are not using enough data points".

5 Graph theory

5.1 VVEEMAD and basic graphs

1. Write down the VVEEMAD for the graphs





(a) (b) (c) (d) (e) (f)

2. Write down the VVEE for the following graphs
3. Describe the six graphs in Q2 using the terminology {tree, line, cycle, complete graph}.
4. Draw the graph from the following V, E
   - $V = \{1, 2, 3, 4\}$, $E = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$
   - $V = \{1, 2, 3, 4\}$, $E = \varnothing$
   - $V = \{1, 2, 3, 4\}$, $E = \{(1, 4), (2, 3), (2, 4), (3, 4)\}$
   - $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4)\}$
5. Draw a graph with 4 vertices that
   - has no edge
   - has 2 edges, and one vertex has degree 2, and the graph is simple graph.
   - has 2 edges, and one vertex has degree 2, and the graph is not simple graph.
6. Draw the complete graph K_4
7. Draw the complete bipartite graph $K_{2,4}$
8. **Terminologies**: explain the following
   - What is a graph?
   - What is the difference between directed graph and undirected graph?
   - What is the difference between simple graph and multigraph?
   - What is a subgraph of a graph?
   - What is a path?
   - What is the length of a path?
   - What is a cycle?
   - What is the difference between tree and forest?
9. Consider a undirected simple graph $G(V, E)$ with $V = \{1, 2, 3, 4, 5\}, E = \{(1, 2), (1, 3), (1, 5), (2, 3), (3, 4), (3, 5), (4, 5)\}$
   - Draw G
   - Find a subgraph of G that is a walk
   - Find a subgraph of G that is a trail
   - Find a subgraph of G that is a path
   - Find a subgraph of G that is a cycle
   - Find all the K_3 (triangle complete graph) subgraph of G
   - Is G a bipartite? If yes, draw G as a bipartite. If not explain why.
10. Consider a undirected simple graph $G(V, E)$ with $V = \{1, 2, 3, 4, 5\}, E = \{(1, 2), (1, 4), (1, 5), (2, 3), (3, 4), (3, 5)\}$
    - Draw G
    - Find a subgraph of G that is a walk
    - Find a subgraph of G that is a trail
    - Find a subgraph of G that is a path
    - Find a subgraph of G that is a cycle
    - Find all the K_3 (triangle complete graph) subgraph of G
    - Is G a bipartite? If yes, draw G as a bipartite. If not explain why.

5.2 Mathematics of graph theory

1. A many different graphs with n vertices are there?
2. Consider a simple undirected graph $G(V, E)$ with $|V| > 2$. Prove that for any G, there are two vertices share the same degree.
   Hint: proof by contradiction.
3. Given a incidence matrix $\boldsymbol{M}$, what does the sum of elements in every row mean?
4. Given a incidence matrix $\boldsymbol{M}$, what does the sum of elements in every column mean?
5. Given a undirected graph G and its adjacency matrix $\boldsymbol{A}$. What does the sum of elements in every row of $\boldsymbol{A}$ mean?
6. Given a undirected graph G and its adjacency matrix $\boldsymbol{A}$. What does the sum of elements in every column of $\boldsymbol{A}$ mean?
7. Given a directed graph G and its incidence matrix $\boldsymbol{M}$. If the diagonal elements of $\boldsymbol{M}$ are all zeros, what does it mean for G?
8. Given a graph G and its adjacency matrix $\boldsymbol{A}$. If the diagonal elements of $\boldsymbol{A}$ are all zeros, what does it mean for G?
9. Given a graph G and its adjacency matrix $\boldsymbol{A}$. If $\boldsymbol{A}$ is symmetric, i.e., $\boldsymbol{A}^\top = \boldsymbol{A}$ where $\boldsymbol{A}^\top$ is the transpose of $\boldsymbol{A}$. What does it mean for G?
10. The complete bipartite graph $K_{m,n}$ is defined by taking two disjoint sets, V_1 of size m and V_2 of size n, and putting an edge between u and v whenever $u \in V_1$ and $v \in V_2$.
    - How many edges does $K_{m,n}$ have?

5.3 Chromatic polynomial

1. Now given $t \in \mathbb{N}$ colors. Write down the chromatic polynomial that represents the number of ways to properly color the following graphs.

(a) (b) (c) (d) (e) (f)

5.4 Solution to VVEEMAD and basic graphs

1. The three graphs share the same $V, |V|$ and $|E|$ as $V = \{1, 2, 3, 4, 5\}$, $|V| = 5$ and $|E| = 4$.

   For (a), $E = \Big\{(1,2),(1,3),(1,4),(2,5)\Big\}$, $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = A, \quad D = \begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$

   For (b), $E = \Big\{(1,4),(2,5),(3,5),(4,5)\Big\}$, $M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} = A, \quad D = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}$

   For (c), $E = \Big\{(1,4),(2,5),(3,4),(4,5)\Big\}$, $M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = A, \quad D = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 3 & \\ & & & & 2 \end{bmatrix}$

   These are the same graph and hence these are isomorphic representations of the same graph.

2. The six graphs share the same $V, |V|$ as $V = \{1,2,3,4\}$, $|V| = 4$.

   For (a), $E = \Big\{(1,2),(2,3),(2,4)\Big\}$, $|E| = 3$.

   For (b), $E = \Big\{(1,2),(2,3),(3,4)\Big\}$, $|E| = 3$.

   For (c), $E = \Big\{(1,2),(2,3),(2,4),(3,4)\Big\}$, $|E| = 4$.

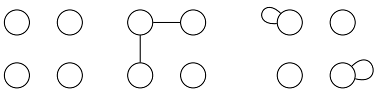
   For (d), $E = \Big\{(1,2),(1,4),(2,3),(3,4)\Big\}$, $|E| = 4$.

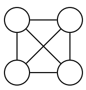
   For (e), $E = \Big\{(1,2),(1,4),(1,3),(2,3),(3,4)\Big\}$, $|E| = 5$.

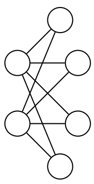
   For (f), $E = \Big\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\Big\}$, $|E| = 6$.

3. For the six graphs
   - (a) is a tree
   - (b) is a line, a tree
   - (c) contains a cycle
   - (d) is a cycle
   - (e) contains three cycles
   - (f) is K_4 (complete graph)

4. The answer is
   - Figure (a) in Q2
   - Draw four dots with no edge
   - Figure (b) in Q2
   - Figure (e) in Q2

5. A simple graph with 4 vertices 

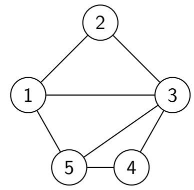
6. K_4 

7. $K_{2,4}$ 

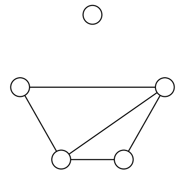
8. **Terminologies**: explain the following
   - A set G of vertex set V and edge set E
   - $(i,j) = (j,i)$ for undirected graph, $(i,j) \neq (j,i)$ for directed graph
   - For simple graph only one (i,j), for multigraph, multiple (i,j) are possible
   - A subset $S(U,F)$ of a graph $G(V,E)$ is that $U \subset V$ and $F \subset E$.
   - A path = a walk that uses no edge more than once
   - Length of a path = number of edge in the path
   - A cycle = a closed path
   - A tree = an acyclic connected graph. A forest relax the connectivity condition of tree.

9. Consider a undirected simple graph $G(V,E)$ with $V = \{1,2,3,4,5\}, E = \{(1,2),(1,3),(1,5),(2,3),(3,4),(3,5),(4,5)\}$

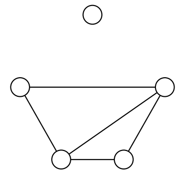
Draw G



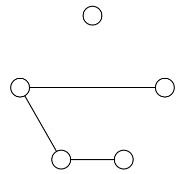
Find a subgraph of G that is a walk
e.g. $3 \to 1 \to 5 \to 4 \to 3 \to 5 \to 1$ (node 1,5 are repeated, edge $(1,5)$ is repeated)
Note: there are more than 1 solution and this is one example of sol.



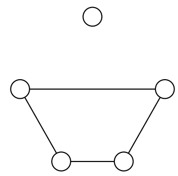
Find a subgraph of G that is a trail
e.g. $3 \to 1 \to 5 \to 4 \to 3 \to 5$ (node 1,5 are repeated, no repeated edge)
Note: there are more than 1 solution and this is one example of sol.



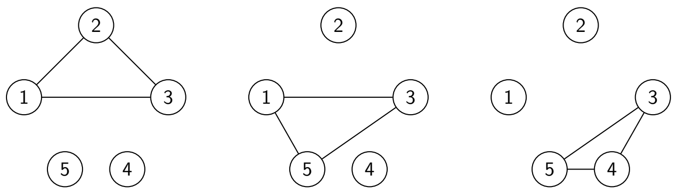
Find a subgraph of G that is a path
e.g. $3 \to 1 \to 5 \to 4$ (no repeated node and edge)
Note: there are more than 1 solution and this is one example of sol.



Find a subgraph of G that is a cycle
e.g. $3 \to 1 \to 5 \to 4 \to 3$ (no repeated node and edge except one node 3)
Note: there are more than 1 solution and this is one example of sol.



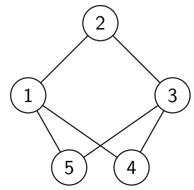
Find all K_3 in G



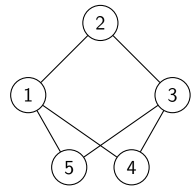
G contains odd cycles (triangle here), so G is not bipartite.

10. Consider a undirected simple graph $G(V,E)$ with $V = \{1,2,3,4,5\}, E = \{(1,2),(1,4),(1,5),(2,3),(3,4),(3,5)\}$

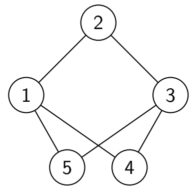
Draw G



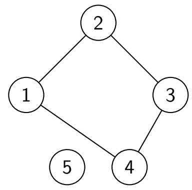
Find a subgraph of G that is a walk
e.g. $1 \to 2 \to 3 \to 4 \to 1 \to 5 \to 3 \to 4$
Note: there are more than 1 solution and this is one example of sol.



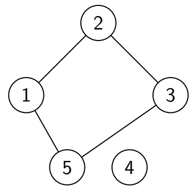
Find a subgraph of G that is a trail
e.g. $1 \to 2 \to 3 \to 4 \to 1 \to 5 \to 3$
Note: there are more than 1 solution and this is one example of sol.



Find a subgraph of G that is a path
e.g. $1 \to 2 \to 3 \to 4 \to 1$
Note: there are more than 1 solution and this is one example of sol.
This path is also a cycle



Find a subgraph of G that is a cycle
e.g. $1 \to 2 \to 3 \to 5 \to 1$
Note: there are more than 1 solution and this is one example of sol.



There is no K_3 in G.

G is bipartite. Furthermore, G is $K_{2,2}$ (complete 2-3 bipartite): the two groups of nodes $\{1, 3\}$ and $\{5, 2, 4\}$

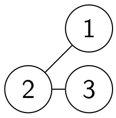
5.5 Solution to mathematics of graph theory

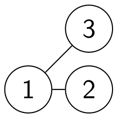
1. $2^{\binom{n}{2}}$
2. For a simple graph with $|V| = n$ nodes, the degree of all the nodes are within in $0, 1, 2, ..., n-1$. (Here the -1 in n refers to no self-loop). Now we prove the statement using proof by contradiction.
   - Assume the statement is false. I.e., there is no two nodes share the same degree. That means all the nodes have distinct degree.
   - Since we have n nodes and we have n possible degree $0, 1, ..., n-1$. That means one node (call it u) has degree 0 and one node (call it v) has degree $n-1$.
   - (u, v) brings a contradiction: $\deg(u) = 0$ means u does not connect with everybody, $\deg(v) = n-1$ means v connects with everybody, including u.
   - Therefore, the assumption is false, and there are two nodes sharing the same degree in a simple graph.
3. The number of other nodes this node has pointed to.
4. The number of nodes pointed to this node.
5. The degree of this node.
6. The degree of this node.
7. The graph has no self-loop.
8. The graph has no self-loop.
9. The graph is undirected.
10. mn

5.6 Solution to chromatic polynomial

1. Let $\chi(t)$ denotes the chromatic polynomial.
   - $\chi_a(t) = t^4 - 3t^3 + 3t^2 - t = t(t-1)^3$
     Explanation: top node have t ways to color.
     Central node have $t-1$ ways to color, here -1 means do not share the same color as the top node.
     For the bottom two nodes, they only need to avoid having the same color as the central node, hence they both have $t-1$ ways to color.

- $\chi_b(t) = t^3 - 2t^2 + t = t(t-1)^2$

  Explanation 1 : consider labeling 
  Node 1 have t ways to color.
  Node 2 have $t-1$ ways to color, here -1 means do not share the same color as the top node.
  Node 3 needs to avoid having the same color as the central node, hence it has $t-1$ ways to color.

  Explanation 2 : consider labeling 
  Node 1 have t ways to color.
  Node 2 and node 3 have $t-1$ ways to color, however we have two cases

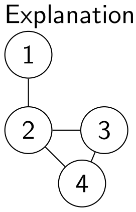
  - case 1. Node 3 has the same color as node 2
    Node 2 has $t-1$ ways to color
    Node 3 only has 1 ways to color (because it has to be the same as node 2)
  - case 2. Node 3 has different color as node 2
    Node 2 has $t-1$ ways to color
    Node 3 has $t-2$ ways to color (because it has to be different from node 1 and node 2)

  So we have

$$\underbrace{t}_{\text{node 1}} \underbrace{\cdot}_{\text{product rule}} \Big(\underbrace{(t-1) \cdot 1}_{\text{case 1}} \underbrace{+}_{\text{sum rule}} \underbrace{(t-1) \cdot (t-2)}_{\text{case 2}} \Big) = t(t-1)^2$$

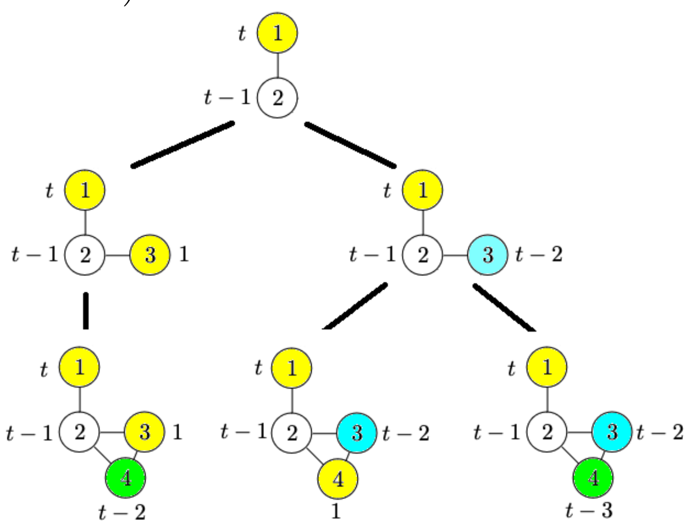
- $\chi_c(t) = t^3 - 3t^2 + 2t = t(t-1)(t-2)$
  Explanation: bottom node have t ways to color.
  Top left node have $t-1$ ways to color, here -1 means do not share the same color as the bottom node.
  Top right node, it has to avoid the two colors of other nodes, hence it has $t-2$ ways to color.
- $\chi_d(t) = t^4 - 2t^3 + t^2 = t^2(t-1)^2$
  Explanation: the isolated node has no constraint on coloring, so it has t ways to color. The remaining 3 nodes is case (b) hence $\chi_b(t) = t(t-1)^2$, By product rule, $\chi_d(t) = t \cdot \chi_b(t) = t^2(t-1)^2$
- $\chi_e(t) = t^4 - 6t^3 + 11t^2 - 6t = t(t-1)(t-2)(t-3)$
- $\chi_f(t) = t^4 - 4t^3 + 5t^2 - 2t = t(t-1)\Big(t-2+t-2+(t-2)(t-3)\Big)$

  Explanation: consider

  1 has t ways to color
  2 has $t-1$ ways to color
  3 has two cases

  - case 1: same as node 1, so it has 1 way
    node 4 has $t-2$ ways
  - case 2: different from node 1, so it has $t-2$ (different from node 1 and node 2)
    node 4 also has two cases
    case 1 same as node 1, so it has 1 way
    case 2 different from node 1, so it has has $t-3$ (different from node 1,2,3)

$$t(t-1)\Big(1 \cdot (t-2) + (t-2) \cdot 1 + (t-2)(t-3)\Big)$$

6 Difficult combinatorics problems

Here

- AHSME = American High School Mathematics Examination
- IMO = International Mathematical Olympiad

6.1 Problems

1. An integer is called square-free if it is not divisible by the square of an integer larger than 1. For example, 9 is not square-free because $9 = 3^2$, 28 is not square free because $28 = 7 \cdot 2^2$, and $11, 15$ are square-free. How many integers between 1 and 100 inclusive are square-free?

2. If α is a real number and $n \geq 1$ is a whole number, show that there is a rational number p/q so that
$$|\alpha - \frac{p}{q}| < \frac{1}{nq}.$$

3. **[China 1990]** In an arena, each row has 199 seats. One day, 1990 students are coming to attend a soccer match. IT is only know that at most 39 students are from the same school. If students from the same school must sit in the same row, find the minimum number of rows that must be reserved for these students.

4. **[HSM 1994]** Find the number of subsets of $\{1, 2, ..., 2000\}$, the sum of whose elements is divisible by 5.

5. **[AHSME 1998]** A 7-digit telephone number $d_1d_2d_3 - d_4d_5d_6d_7$ is called memorable if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be any of the ten decimal digits $0, 1, 2, ..., 9$, find the number of different memorable telephone numbers.

6. **[AHSME 1994]** Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?

7. **[IMO Shortlist 1994]** A subset M of $\{1, 2, 3, ..., 15\}$ does not contain three elements whose product is a perfect square. Determine the maximum number of elements in M

8. **[IMO2022]**. The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and moves all coins in that chain to the left end of the row.

   For example, if $n = 4$ and $k = 4$, the process starting from the ordering AABBBABA would be
$$AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to BBBBAAAA \to \cdots$$

   Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

6.2 Solution

1. Let $N = \{1, 2, ..., 100\}$. Recall that the number of integers in N divisible by 2 is $\lfloor \frac{100}{2} \rfloor$. Hence, the number of integers in N that are divisible by 2^2 is $\lfloor \frac{100}{2^2} \rfloor$, this is the number of integers that are not 2-square-free, i.e., these numbers can be written as $k \cdot 2^2$ for some k. Then we have
$$\begin{aligned} s_1 &= \left\lfloor \frac{100}{2^2} \right\rfloor + \left\lfloor \frac{100}{3^2} \right\rfloor + \left\lfloor \frac{100}{5^2} \right\rfloor + \left\lfloor \frac{100}{7^2} \right\rfloor + \underbrace{\left\lfloor \frac{100}{11^2} \right\rfloor}_{=0} + 0 + \cdots = 42 \\ s_2 &= \left\lfloor \frac{100}{2^2 3^2} \right\rfloor + \left\lfloor \frac{100}{2^2 5^2} \right\rfloor + \underbrace{\left\lfloor \frac{100}{2^2 7^2} \right\rfloor}_{=0} + 0 + \cdots = 3 \end{aligned}$$

   The solution, by inclusion-exclusion principle and complement
$$100 - 42 + 3 = 61.$$

2. We will need the notion of the *fractional part* of a real number. The fractional part of a number is the part to the right of the decimal point. Given any real number x, let $[x]$ be the fractional part of x. For example, $\pi = 3.14159...$ and its fractional part is $[\pi] = 0.14159....$

   The two important properties of the fractional part of a number are

   - $0 \leq \{x\} < 1$ for any x
   - $x - [x]$ is always a whole number.

Now consider $n+1$ real numbers live in the interval $[0,1)$ as

$$[0\alpha],[\alpha],[2\alpha],....,[n\alpha].$$

Suppose we break this interval up into n pieces

$$[0,1/n),[1/n,2/n),...,[(n-1)/n,1).$$

Then each of the $n+1$ numbers must belong to one of these pieces, hence by the pigeonhole principle there must be an interval with at least two different numbers in it. That is, for some $m \geq 0$ and $q > 1$ we have both $[m\alpha]$ and $[(m+q)\alpha]$ in the same interval of length $1/n$. But then their difference $(m+q)a - m\alpha = q\alpha$ has fractional part in $[0,1/n)$. Let p be the integer $p = q\alpha - [q\alpha]$. Then rearranging we have

$$\big|q\alpha - p\big| = \big|[q\alpha]\big| < \frac{1}{n}.$$

Dividing both sides of this inequality by the positive integer q gives us

$$\big|\alpha - \frac{p}{q}\big| < \frac{1}{nq}.$$

3. Consider 200. Its largest divisor not exceeding 39 is 25. Note that $1990 = 79 \times 25 + 15$. If 79 schools send 25 students each and one school sends 15 student, it will take at least $\lceil 79/\lfloor 199/26 \rfloor \rceil = 12$ rows to seat all the students.

   WE now show that 12 rows are enough. Start seating the students school by school and row by row, filling all the seats of the first 10 rows, even if students from some schools are split between two rows. This can happen to at most 9 schools. Remove the students from those schools and pack them into two rows. This is possible since each row can hold students from at least 5 schools as $5 \times 39 = 195 < 199$.

4. Consider the polynomial
$$f(x) = (1+x)(1+x^2)\cdots(1+x^{2000}).$$
   There is a bijection between each subset $\{a_1, a_2, ..., a_m\}$ of $\{1, 2, ..., 2000\}$ and the term $x^{a_1}x^{a_2}\cdots x^{a_m}$. Hence we are looking for the sum of coefficients of terms x^{5k} in $f(x)$, for k being positive integers. Let S denote that sum.

   (Introduction of complex number) Let $\xi = e^{2\pi i/5}$ be a 5^{th} root of unity. Then $\xi^5 = 1$ and $1+\xi+\xi^2+\xi^3+\xi^4 = 0$. Hence
$$S = \frac{1}{5}\sum_{j=1}^{5} f(\xi^j).$$
   Note that $\xi, \xi^2, \xi^3, \xi^4, \xi^5 = 1$ are roots of $g(x) = x^5 - 1$, that is
$$g(x) = x^5 - 1 = (x-\xi)(x-\xi^2)(x-\xi^3)(x-\xi^4)(x-\xi^5).$$
   Then
$$g(-1) = -2 = (-1-\xi)(-1-\xi^2)(-1-\xi^3)(-1-\xi^4)(-1-\xi^5).$$
   Therefore
$$(1+\xi)(1+\xi^2)(1+\xi^3)(1+\xi^4)(1+\xi^5) = 2$$
   and $f(\xi) = 2^{400}$. Similarly, $f(\xi^j) = 2^{400}$ for $j = 2,3,4$. Lastly $f(\xi^5) = f(1) = 2^{2000}$, hence
$$S = \frac{1}{5}\Big(4\cdot 2^{400} + 2^{2000}\Big) = \frac{2^{402}+2^{2000}}{5}.$$

5. Let A be the set of telephone numbers for which $d_1d_2d_3$ is the same as $d_4d_5d_6$, and let B be the set of telephone numbers for which $d_1d_2d_3$ is the same as $d_5d_6d_7$. A telephone number $d_1d_2d_3 - d_4d_5d_6d_7$ belongs to $A \cap B$ if $d_1 = d_4 = d_5 = d_2 = d_6 = d_3 = d_7$, i.e., the whole number is a constant string (all digits are the same) Hence $|A \cap B| = 10$ because there are ten different constant strings. By the Imclusion-Exclusion Principle,
$$|A \cup B| = |A| + |B| - |A \cap B| = 10^3 \cdot 1 \cdot 10 + 10^3 \cdot 10 \cdot 1 - 10 = 19990.$$

6. Imagine the six students standing in a row before they are seated. There are 5 spaces between them, each of which may be occupied by at most one of the 3 professors. Therefore, there are $5 \cdot 4 \cdot 3 = 60$ ways the professors can select their places.

7. Let $S = \{1, 2, 3, ..., 15\}$. We call a set $M \subset S$ is *good* if M does not contain three elements whose product is a perfect square. We call a triple of numbers $\{i, j, k\}$, $1 \leq i < j < k \leq 15$ *bad* if ijk is a perfect square.

   Let m denotes the maximum value of $|M|$ where M is good.

- We show $m \leq 11$.
Since there are disjoint bad triples $B_1 = \{1, 4, 9\}, B_2 = \{2, 6, 12\}, B_3 = \{3, 5, 15\}, B_4 = \{7, 8, 14\}$, if $|M| = 12$, all three numbers in at least on of these triples are in M by pigeon-hole principle. Hence M is bad if $|M| \geq 12$ and we conclude that $m \leq 11$.

- Now we discuss what if $m = 11$.
Let M be a good set with $m = |M| = 11$. Then $M = S \setminus \{a_1, a_2, a_3, a_4\}$, where $a_i \in \boldsymbol{B}_i$ for $i = 1, 2, 3, 4$. Hence $10 \in M$.
Since $10 \in M$ and $B_1 = \{1, 4, 9\}, B_4 = \{7, 8, 14\}, B_5 = \{2, 5, 10\}, B_6 = \{6, 15, 10\}$ are bad triples with 10 as the only repeated element, $M = S \setminus \{b_1, b_4, b_5, b_6\}$ where
  - $b_1 \in B_1$
  - $b_4 \in B_4$
  - $b_5 \in \{2, 5\}$
  - $b_6 \in \{6, 15\}$

  Therefore $\{3, 12\} \subset M$. Then 1,4,9 are not in M. Since there are still two disjoint bad triples $\{2, 3, 6\}$ and $\{7, 8, 14\}$, we need to delete at least two more numbers to make M good. Hence $|M| \leq 10$, which contradicts with the assumption that $|M| = 11$. Hence our assumption was wrong and $m \leq 10$.

- The set $\{1, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$ satisfies the conditions of the problem. Hence 10 is the maximum number of elements of M.

8. `https://www.youtube.com/watch?v=KHn3xD6wS3A`

END OF DOCUMENT (ver. January 15, 2024)