Projection operator is non-expansive
and also firmly non-expansive

Andersen Ang

Mathématique et recherche opérationnelle, UMONS, Belgium

mashun.ang@umons.ac.be   Homepage: angms.science

First draft: November 21, 2018
Last update: December 22, 2020
Euclidean projection

▶ Given a closed, non-empty, convex set $C \subset \mathbb{R}^n$, and a point $z \in \mathbb{R}^n$ ($z$ can be in or outside $C$), the Euclidean projection $P$ produce a point $x$ that

$$x = P(z) = \arg\min_{x \in C} \frac{1}{2} \|x - z\|^2_2.$$

i.e., input of $P$ is a point $z$, the output of $P$ is a point $x \in C$ that is closest to $z$.

▶ This problem always has a unique solution because of Weierstrass’s theorem. Details here.

▶ Here we consider Euclidean projection. Example of other projection: Bregman projection.
Euclidean projection

\[ x = P(z) = \arg\min_{x \in C} \frac{1}{2} \|x - z\|^2. \]

- Two possibilities between \( z \) and \( C \):
  - \( z \in C \).
    Projection becomes *Identity operator*: it output \( x \) as \( z \)
    \[ P(z) = z. \]
  - \( z \notin C \).
    Projection finds the closest point \( x \in C \) to \( z \).
    - Distance here is defined in terms of Euclidean distance
      \[ \|v\|_2 = \sqrt{v_1 + v_2 + \cdots + v_n}. \]
    - A property of \( P \): \( x \) will be located on the boundary of \( C \): \( x \in \text{bd}C \).
Bourbaki-Cheney-Goldstein inequality\(^2\)

- **Modern names:** Obtuse angle criterion, Projection theorem\(^1\).

- **What is it:** a variational characterization of projection operator

\[
\langle \mathbf{z} - P(\mathbf{z}), \mathbf{x} - P(\mathbf{z}) \rangle \leq 0, \quad \forall \mathbf{x} \in C.
\]

The angle in between is obtuse ($\theta \geq 90^\circ$).

---

\(^1\) Note the name “projection theorem” is usually refer to projection in the context of Hilbert space (= vector space equipped with inner product, an operation that allows defining lengths and angles.)

Why \( \langle z - P(z), x - P(z) \rangle \leq 0, \; \forall x \in C \). is true

▶ First, the line segment \( L_\lambda : (1 - \lambda)P(z) + \lambda x \) for \( \lambda \in [0, 1] \) is always inside \( C \).

▶ Next, let \( z - L_\lambda \) be the distance between \( z \) and the line \( L_\lambda \).

▶ As \( L_\lambda \in C \), so we want \( z \) to be close to \( L_\lambda \) as close as possible. That is, we want to minimize distance(\( z, L_\lambda \)):

\[
\phi(\lambda) = \text{distance}(z, L_\lambda) := \left\| z - \left( (1 - \lambda)P(z) + \lambda x \right) \right\|^2_2.
\]

Expand \( \phi(\lambda) \) gives an quadratic function on \( \lambda \). Such parabola has a vertex, minimizing \( \phi \) at the vertex gives the obtuse angle criterion.
We have \( \phi(\lambda) = a\lambda^2 + b\lambda + c \)

\[
\phi(\lambda) = \left\| z - \left( (1 - \lambda)P(z) + \lambda x \right) \right\|^2_2 \\
= \left\| \left( z - P(z) \right) - \lambda \left( x - P(z) \right) \right\|^2_2 \\
= \left\| z - P(z) \right\|^2_2 - 2\left\langle z - P(z), x - P(z) \right\rangle \lambda + \left\| x - P(z) \right\|^2_2 \lambda^2.
\]

The vertex of \( a\lambda^2 + b\lambda + c \) is at \( \lambda = \frac{-b}{2a} \)

\[
\lambda = -\frac{-2\left\langle z - P(z), x - P(z) \right\rangle}{2\left\| x - P(z) \right\|^2_2} = \frac{\left\langle z - P(z), x - P(z) \right\rangle}{\left\| x - P(z) \right\|^2_2}.
\]

We want \( \lambda \) to be non-positive: so that on the line segment \( L_\lambda \), the parabola is minimized at 0. \( \lambda \) is non-positive gives
\[
\left\langle z - P(z), x - P(z) \right\rangle \leq 0.
\]
(note: the inner product \( \langle z - P(z), x - P(z) \leq 0 \) doesn’t need to be negative, only non-positive.)
Projection operator is non-expansive

- A function $f$ is called non-expansive\(^3\) if $f$ is $L$-Lipschitz with $L \leq 1$. That is, for any two points $x, z \in \text{dom } f$,

$$\|f(x) - f(z)\| \leq L\|x - z\|,$$

where $L \leq 1$.

It means the distance between the mapped points is smaller than that of the unmapped points in the original space.

- Projection operator is non-expansive:

$$\|P(x) - P(z)\|_2 \leq \|x - z\|_2.$$

- Next slide: variational characterization implies non-expansiveness. i.e.,

$$\langle z - P(z), x - P(z) \rangle \leq 0 \ \forall x \in C \implies \|P(x) - P(z)\|_2 \leq \|x - z\|_2.$$

\(^3\)Non-expansive becomes contractive if $L < 1$. 

7 / 9
Begins with the variational characterization

\[ \langle z - P(z) , x - P(z) \rangle \leq 0 \ \forall x \in C. \quad (1) \]

replace \( x \) by \( P(x) \) in (1)

\[ \langle z - P(z) , P(x) - P(z) \rangle \leq 0. \quad (2) \]

replace \( z \) by \( x \) and \( x \) by \( P(z) \) in (1)

\[ \langle x - P(x) , P(z) - P(x) \rangle \leq 0. \quad (3) \]

(2)+(3) will cancel \( P(z) - P(x) \), not good. So flip the sign of (3) gives

\[ \langle P(x) - x , P(x) - P(z) \rangle \leq 0. \quad (4) \]

(2) + (4) \[ \langle z - P(z) + P(x) - x , P(x) - P(z) \rangle \leq 0 \]

\[ \iff \langle z - x + P(x) - P(z) , P(x) - P(z) \rangle \leq 0 \]

\[ \iff \langle z - x , P(x) - P(z) \rangle \leq - \langle P(x) - P(z) , P(x) - P(z) \rangle \]

\[ \iff \langle z - x , P(z) - P(x) \rangle \geq \| P(x) - P(z) \|_2^2 \]

\[ \iff \| (z - x)^\top (P(z) - P(x)) \|_2 \geq \| P(x) - P(z) \|_2^2 \]

By Cauchy-Schwarz inequality, the left-hand-side is upper bounded by

\[ \| z - x \|_2 \| P(z) - P(x) \|_2^{\frac{1}{2}} \]

we get \[ \| z - x \|_2 \| P(z) - P(x) \|_2^{\frac{1}{2}} \geq \| P(x) - P(z) \|_2^2 \]

Cancels \( \| P(x) - P(z) \|_2^{\frac{1}{2}} \) finishes the proof.
Projection operator is firmly non-expansive

▶ In fact, projection operator is not just non-expansive, but also **firmly non-expansive** (stronger condition).

▶ A definition\(^4\) of firmly non-expansive is

\[
\langle (x - P(x)) - (z - P(z)), P(x) - P(z) \rangle \geq 0.
\]

In fact, we have this already: from the proof of non-expansiveness, (2)+(4) gives

\[
\langle z - P(z) + P(x) - x, P(x) - P(z) \rangle \leq 0,
\]

we get firmly non-expansiveness by flipping the inequality sign and the sign of the red terms.

---