The recursion $x^{k+1} = \frac{\rho_k}{2}T(x^k) + \left(1 - \frac{\rho_k}{2}\right)x^k$ converges to a fixed point of T if map T is non-expansive

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First draft: August 15, 2019
Last update : October 16, 2019

A theorem on fixed point convergence

A mapping : Let $T : \mathbb{R}^m \to \mathbb{R}^n$ is a non-expansive map. i.e., for all $x, y \in \text{dom}T = \mathbb{R}^m$, we have

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

A sequence : Consider a sequence $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 2)$ with $\inf_k\{\rho_k\} > 0$ and $\sup_k\{\rho_k\} < 2$.

A theorem on recursion : If x_k is generated by the recursion

$$x_{k+1} = \frac{\rho_k}{2} T(x_k) + \left(1 - \frac{\rho_k}{2}\right) x_k,$$

then the sequence $\{x_k\}_{k \in \mathbb{N}}$ converges to a fixed point of T (provided that T has a fixed point).

This document : on the proof of this theorem.

Remarks

- T is non-expensive means T is Lipschitz (with $L = 1$)
- $\rho_0, \rho_1, \rho_2, \ldots$ are a (possibly non-constant) sequence, bounded between (0,2), and they never touch 0 nor 2.
- $x = \mathrm{Id}(x) = Ix$, so $\frac{\rho_k}{2} T\big(x_k\big) + \big(1 + \frac{\rho_k}{2}\big)\, x_k$ is a linear combination of the operator T and the Identity operator Id.
- Definition of fix point : v is a fixed point of T if $T(v) = v$.

The idea of the proof

To prove the statement, we show two things

- Sufficient decrease property : suppose x^* is a fixed point of T,

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \text{something positive}, \forall k$$

- Show the limit point of $\{x_k\}_{k \in \mathbb{N}}$ is the fixed point of T

The proof ... (1/5)

We begins with $\|x_{k+1} - x^*\|^2$: by the definition of the recursion, we have

$$\|x_{k+1} - x^*\|^2 = \left\| \frac{\rho_k}{2} T(x_k) + \left(1 - \frac{\rho_k}{2}\right) x_k - x^* \right\|^2$$

Trick

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \left\| \frac{\rho_k}{2} T(x_k) + \left(1 - \frac{\rho_k}{2}\right) x_k - x^* + \frac{\rho_k}{2} x^* - \frac{\rho_k}{2} x^* \right\|^2 \\ &= \left\| \frac{\rho_k}{2} \Big(T(x_k) - x^*\Big) + \left(1 - \frac{\rho_k}{2}\right) (x_k - x^*) \right\|^2 \end{aligned}$$

Expand

$$\|x_{k+1} - x^*\|^2 = \begin{array}{c} \left(\frac{\rho_k}{2}\right)^2 \left\|T(x_k) - x^*\right\|^2 + \left(1 - \frac{\rho_k}{2}\right)^2 \|x_k - x^*\|^2 \\ +2\,\frac{\rho_k}{2} \left(1 - \frac{\rho_k}{2}\right) \Big\langle T(x_k) - x^*, x_k - x^* \Big\rangle \end{array}$$

Do not simplify $2\,\frac{\rho_k}{2}$ to ρ_k (see next slide).

Continue from last page, trick again

$$\begin{aligned}\|x_{k+1}-x^*\|^2 \;=\; & \left(\frac{\rho_k}{2}\right)^2 \left\|T(x_k)-x^*\right\|^2 + \left(1-\frac{\rho_k}{2}\right)^2 \|x_k-x^*\|^2 \\ & +2\,\frac{\rho_k}{2}\left(1-\frac{\rho_k}{2}\right)\left\langle T(x_k)-x^*, x_k-x^*\right\rangle \\ & +\frac{\rho_k}{2}\left(1-\frac{\rho_k}{2}\right)\|T(x_k)-x_k\|^2 \\ & -\frac{\rho_k}{2}\left(1-\frac{\rho_k}{2}\right)\|T(x_k)-x_k\|^2\end{aligned}$$

Furthermore

$$\begin{aligned}\|T(x_k)-x_k\|^2 &= \|T(x_k)-x^*-(x_k-x^*)\|^2 \\ &= \|T(x_k)-x^*\|^2 - 2\langle T(x_k)-x^*, x_k-x^*\rangle + \|x_k-x^*\|^2\end{aligned}$$

We have (after cancelling terms)

$$\begin{aligned}\|x_{k+1}-x^*\|^2 \;=\; & \frac{\rho_k}{2}\left\|T(x_k)-x^*\right\|^2 + \left(1-\frac{\rho_k}{2}\right)\|x_k-x^*\|^2 \\ & -\frac{\rho_k}{2}\left(1-\frac{\rho_k}{2}\right)\|T(x_k)-x_k\|^2\end{aligned}$$

The proof . . . (3/5)

Continue, as x^* is a fixed point of T so $x^* = T(x^*)$ and

$$\|x_{k+1} - x^*\|^2 = \frac{\rho_k}{2} \left\|T(x_k) - T(x^*)\right\|^2 + \left(1 - \frac{\rho_k}{2}\right) \|x_k - x^*\|^2 - \frac{\rho_k}{2} \left(1 - \frac{\rho_k}{2}\right) \|T(x_k) - x_k\|^2$$

As T is non-expansive

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq \frac{\rho_k}{2} \|x_k - x^*\|^2 + \left(1 - \frac{\rho_k}{2}\right) \|x_k - x^*\|^2 - \frac{\rho_k}{2} \left(1 - \frac{\rho_k}{2}\right) \|T(x_k) - x_k\|^2 \\ \text{combine terms} &= \|x_k - x^*\|^2 - \frac{\rho_k}{2} \left(1 - \frac{\rho_k}{2}\right) \|T(x_k) - x_k\|^2 \end{aligned}$$

As $\{\rho_k\}_{k\in\mathbb{N}} \subset (0, 2)$, the term $\epsilon_k := \frac{\rho_k}{2} \left(1 - \frac{\rho_k}{2}\right) > 0$ and

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \underbrace{\epsilon_k \|T(x_k) - x_k\|^2}_{\text{something positive}}.$$

We proved the sufficient decrease part.

The proof ... (4/5)

Telescoping : consider $k = 0$ to $k = K - 1$

$$
\begin{array}{rlcl}
k = K-1 & \|x_K - x^*\|^2 & \leq & \|x_{K-1} - x^*\|^2 - \epsilon_{K-1}\|T(x_{K-1}) - x_{K-1}\|^2 \\
k = K-2 & \|x_{K-1} - x^*\|^2 & \leq & \|x_{K-2} - x^*\|^2 - \epsilon_{K-2}\|T(x_{K-2}) - x_{K-2}\|^2 \\
& \vdots & & \vdots \\
k = 1 & \|x_2 - x^*\|^2 & \leq & \|x_1 - x^*\|^2 - \epsilon_1\|T(x_1) - x_1\|^2 \\
k = 0 & \|x_1 - x^*\|^2 & \leq & \|x_0 - x^*\|^2 - \epsilon_k\|T(x_0) - x_0\|^2
\end{array}
$$

Sum all of them

$$\|x_K - x^*\|^2 \leq \|x_0 - x^*\| - \sum_{k=0}^{K-1} \epsilon_k \|T(x_k) - x_k\|^2$$

As all $\epsilon_k > 0$, there exist some $\epsilon > 0$ that

$$\|x_K - x^*\|^2 \leq \|x_0 - x^*\| - \epsilon \sum_{k=0}^{K-1} \|T(x_k) - x_k\|^2$$

From

$$\|x_K - x^*\|^2 \leq \|x_0 - x^*\| - \epsilon \sum_{k=0}^{K-1} \|T(x_k) - x_k\|^2,$$

- $\{x_k\}_{k\in\mathbb{N}}$ is a bounded sequence
- $\{\|T(x_k) - x_k\|^2\}_{k\in\mathbb{N}}$ is summable so $\lim_{k\to\infty} T(x_k) - x_k = 0$.
- **Existence of a limit point for $\{x_k\}_{k\in\mathbb{N}}$**
  As $\{x_k\}_{k\in\mathbb{N}}$ is a bounded sequence, it has at least one limit point.
- **The limit point of $\{x_k\}_{k\in\mathbb{N}}$ is the limit point of T**
  Consider a limit point x_∞ and a corresponding subsequence of indices $\mathcal{K}$ such that the subsequence $x_k \to x_\infty$ for $k \in \mathcal{K}$, then by Lipschitz continuity of T we have

  $$T(x_\infty) - x_\infty = \lim_{k\to\infty, k\in\mathcal{K}} T(x_k) - x_k$$

  which will equal to 0 since entire sequence $\{T(x_k) - x_k\}_{k\in\mathbb{N}}$ converges to zero. Hence $T(x_\infty) = x_\infty$.
- We can then put $x_* = x_\infty$ and get $\|x_{k+1} - x_\infty\| \leq \|x_k - x_\infty\|$.

Last page - summary

For a non-expansive map $T : \mathbb{R}^m \to \mathbb{R}^n$, if x_k is generated as

$$x_{k+1} = \frac{\rho_k}{2} T(x_k) + \left(1 - \frac{\rho_k}{2}\right) x_k$$

under a sequence $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 2)$, then the sequence $\{x_k\}_{k \in \mathbb{N}}$ converges to a fixed point of T (provided that T has a fixed point).

A special case of this theorem the average operator

$$x_{k+1} = \frac{1}{2} T(x_k) + \frac{1}{2} x_k.$$

which is when $\rho_k = 1$ for all k.

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