Linear Optimization

A 10 minutes super fast review of key concepts

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LO forms

- Standard form (in matrix-vector notation)

$$\begin{aligned} \max \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- Expanded form

$$\begin{aligned} \max \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{s.t.} \quad & \langle \mathbf{a}^i, \mathbf{x} \rangle = b_i, i \in \mathcal{E} \\ & \langle \mathbf{a}^i, \mathbf{x} \rangle > b_i, i \in \mathcal{I}_1 \\ & \langle \mathbf{a}^i, \mathbf{x} \rangle < b_i, i \in \mathcal{I}_2 \\ & x_i \geq 0, i \in \mathcal{N}_1 \\ & x_i \leq 0, i \in \mathcal{N}_2 \end{aligned}$$

  where $\mathbf{a}^i$ is not power of $\mathbf{a}$, but a label. It means the i-th row of $\mathbf{A}$.
- LO means cost function and constraints are all linear.

Solution of LO

- The most general form of LO

$$\max\langle \mathbf{c}, \mathbf{x}\rangle \text{ s.t. } \mathbf{x} \in P,$$

where the P is the feasible set, a polyhedron described by an system of linear inequality in the form $\mathbf{A}\mathbf{x} \leq \mathbf{b}$

$$P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

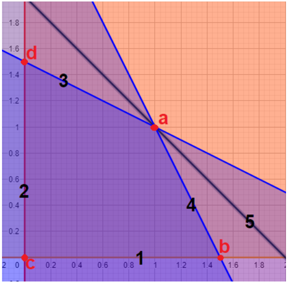
by augmenting $\mathbf{A}, \mathbf{b}$, the inequality $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ can represent both $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ in the standard form of LO.

- There are two cases for the feasible set. Feasible set is non-empty :
  - Exists a unique optimal solution in the set of feasible solutions
  - Multiple optimal solutions in the set of feasible solutions
  - Optimal cost is ∞, no feasible solution is optimal
- Feasible set is empty, no feasible solution.

- The set $p = \{\mathbf{x} \mid \langle \mathbf{a}, \mathbf{x} \rangle = b\}$ is a hyperplane, the normal of p is $\mathbf{a}$. If $b = 0$, $-\mathbf{a}$ is also the normal of p : $\langle \mathbf{a}, \mathbf{x} \rangle = 0 \implies \langle -\mathbf{a}, \mathbf{x} \rangle = 0$.
- The set $\{\mathbf{x} \mid \langle \mathbf{a}, \mathbf{x} \rangle \geq b\}$ is a half-space.
- The intersection of $\underbrace{\text{many}}_{\geq 2}$ half-space is a polyhedron.
- A polyhedron P is a convex set : $\mathbf{x}, \mathbf{y} \in P \implies \lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in P$.
- A point $\mathbf{x} \in P$ is an extreme point of P if it cannot be expressed as a convex combination of any two points in P.
  $\nexists \mathbf{y}, \mathbf{z} \in P \text{ s.t. } \mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}, \lambda \in]0, 1[$.
- Fundamental theorem of LO : if a LO problem has a optimum, the optimum must be one of the extreme point of the feasible set P (as for linear function, maximum or minimum are located at the end)

Illustrating example

$$\begin{aligned} \max \quad & x + y \\ \text{s.t.} \quad & x + 2y \leq 3 \\ & 2x + y \leq 3 \\ & x \geq 0, y \geq 0 \end{aligned}$$



where

- 1 : the line $y = 0$
- 2 : the line $x = 0$
- 3 : the line $x + 2y = 3$
- 4 : the line $2x + y = 3$
- 5 : the line $x + y = 2$

- a : solution of line 3 ∩ line 4
- b : solution of line 1 ∩ line 4
- c : solution of line 1 ∩ line 2
- d : solution of line 2 ∩ line 3

This problem has a global optimum at the point $a(1, 1)$.

Illustrating example – in standard form

$$\begin{aligned} \max \quad & x + y \\ \text{s.t.} \quad & x + 2y \le 3 \\ & 2x + y \le 3 \\ & x \ge 0, y \ge 0 \end{aligned} \qquad \begin{aligned} \max \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$

Or, in a even more compact notation,

$$\begin{aligned} &\max \ \langle \mathbf{c}, \mathbf{x} \rangle \\ &\text{s.t.} \ \mathbf{x} \in P := \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}. \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

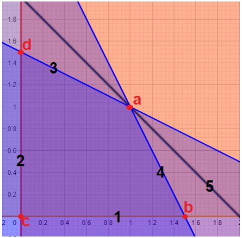
The feasible set is a polyhedron $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}$ with four extreme points : $\{a, b, c, d\}$. We know a is the optimal point (geometrically), but how do we know a is the optimal point algebarically?

Illustrating example - optimal point

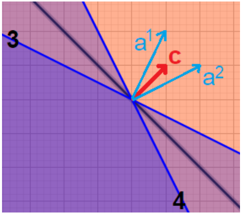
An extreme point is the optimal solution if $\mathbf{c}$ is in the cone defined by the rows of $\mathbf{A}$ that define that extreme point.

$$\max \ \langle \mathbf{c}, \mathbf{x} \rangle \text{ s.t. } \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$



The extreme point a is the optimal sol. as $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the cone defined by the first two rows of $\mathbf{A}$ that define a (corresponds to line 3 and line 4 in the figure). $\mathbf{a}^1 = [1\ 2]$, $\mathbf{a}^2 = [2\ 1]$.

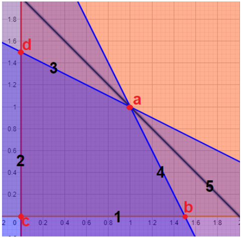


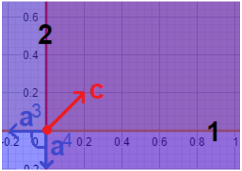
Illustrating example - non-optimal point

The rows of $\mathbf{A}$ that defines c is the last two rows (corresponds to line 1 and line 2 in the figure).

$$\max \ \langle \mathbf{c}, \mathbf{x} \rangle \text{ s.t. } \mathbf{Ax} \le \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$





The vector $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not inside the cone defined by the last two rows of $\mathbf{A}$: $\mathbf{a}^3 = [-1 \ 0]$, $\mathbf{a}^4 = [0 \ -1]$. Therefore point c is not the optimal point.

Conic combination

Algebraically, if a vector $\mathbf{x}$ is inside a cone defined by columns of a matrix $\mathbf{M}$, we have

$$\mathbf{x} = \mathbf{M}\mathbf{y}, \mathbf{y} \geq 0.$$

As $\mathbf{y}$ is nonnegative, it means $\mathbf{x}$ can be written as a conic combination of columns of $\mathbf{M}$.

If $\mathbf{x}$ is inside a cone defined by rows of $\mathbf{M}$, we have

$$\mathbf{x} = \mathbf{M}^\top \mathbf{y}, \mathbf{y} \geq 0.$$

Back to the LO problem, now $\mathbf{c}$ is inside a cone defined by rows of $\mathbf{A}$,

$$\mathbf{c} = \mathbf{A}^\top \mathbf{y}, \mathbf{y} \geq 0.$$

This expression is linked to the duality of LO.

Duality

Primal problem

$$\max \ \langle \mathbf{c}, \mathbf{x} \rangle \text{ s.t. } \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

Dual problem

$$\min \ \langle \mathbf{b}, \mathbf{y} \rangle \text{ s.t. } \ \mathbf{A}^\top \mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0$$

- In the example, $\mathbf{y} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}^\top$ and so $\langle \mathbf{b}, \mathbf{y} \rangle = 2$, which equals to the optimal value of the primal problem. In this case we have strong duality (no duality gap).
- We also have $y_1, y_2 \neq 0$ and $y_3 = y_4 = 0$. The non-zero of $\mathbf{y}$ corresponds to the complementary slackness

$$y_i^* > 0 \implies \langle \mathbf{a}^i, \mathbf{x}^* \rangle = b_i,$$

where $\mathbf{a}^i$ is the i-th row of $\mathbf{A}$.

Complementary Slackness

Primal problem

$$\max \ \langle \mathbf{c}, \mathbf{x} \rangle \text{ s.t.} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

Dual problem

$$\min \ \langle \mathbf{b}, \mathbf{y} \rangle \text{ s.t. } \mathbf{A}^\top \mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0$$

Complementary slackness :

1. if a dual variable is slack, the corresponding primal constraint is tight.
2. if a primal constraint is slack, the corresponding dual variable is tight.

In the LO exmape : $\mathbf{x}^* = [1 \ 1]^\top$, $\mathbf{y}^* = [1/3 \ 1/3 \ 0 \ 0]^\top$.

i	$\langle \mathbf{a}^i, \mathbf{x}^* \rangle \leq b_i$	y_i^*	$y_i^* \geq 0$	primal	dual
1	$=$	$1/3$	$>$	tight	slack
2	$=$	$1/3$	$>$	tight	slcak
3	$<$	0	$=$	slack	tight
4	$<$	0	$=$	slack	tight

Recall : for two scalar α, β, slack means $\alpha < \beta$ or there exists s such that $\alpha + s = \beta$, and tight means $\alpha = \beta$.

Last page - summary

Discussed : the concept of

- Linear optimization problem
- Convex geometry
- Fundamental theorem of LO : if there is a optimal solution, must be at an extreme point
- Duality and complementary slackness

Not discussed : how to solve it

- Simplex method
- Ellipsoid method
- Interior point method

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