Some math of projection $\mathrm{proj}_{U,p}(\boldsymbol{x}) := \underset{\boldsymbol{u}\in U}{\mathrm{argmin}}\ \|\boldsymbol{x}-\boldsymbol{u}\|_p$

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June 8, 2025

1st ver. June 1, 2025

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$\|\boldsymbol{x}\|_p,\ 1 < p < \infty$ is strictly convex

Uniqueness of $\mathrm{proj}_{U,p}$ $\begin{cases} \text{has unique solution} & 1 < p < \infty \\ \text{may has non-unique solution} & p \in \{1, \infty\} \end{cases}$

Continuity of $\mathrm{proj}_{U,p}$ $\begin{cases} \left\| \mathrm{proj}_{U,p}(\boldsymbol{x}) - \mathrm{proj}_{U,p}(\boldsymbol{y}) \right\|_2 \le \gamma \|\boldsymbol{x} - \boldsymbol{y}\|_2 \ \forall \boldsymbol{x}, \boldsymbol{y} & 1 < p < \infty \\ \mathrm{dist}_H\Big(\mathrm{proj}_{U,p}(\boldsymbol{x}),\ \mathrm{proj}_{U,p}(\boldsymbol{y})\Big) \le \gamma \|\boldsymbol{x} - \boldsymbol{y}\|_2 \ \forall \boldsymbol{x}, \boldsymbol{y} & p \in \{1, \infty\} \end{cases}$

Geometry of projection $\mathrm{proj}_{U,p}$: nonempty closed convex polytope

Setup

- $\mathbb{R}^n$: n-dimensional Euclidean space
  - column vectors

- ℓ_p-norm $\|\boldsymbol{x}\|_p : \mathbb{R}^n \to \mathbb{R}$

$$\|\boldsymbol{x}\|_p = \begin{cases} \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{1/p} & 1 \le p < \infty \\ \max\limits_{1 \le i \le n} |x_i| & p = \infty \end{cases}$$

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Geometry of projection $\mathrm{proj}_{U,p}$: nonempty closed convex polytope

$\|\boldsymbol{x}\|_p,\ 1 < p < \infty$ is strictly convex

- We prove strict convexity using strict monotonicity of gradient

$$\langle \nabla \|\boldsymbol{x}\|_p - \nabla \|\boldsymbol{y}\|_p,\ \boldsymbol{x} - \boldsymbol{y} \rangle > 0 \ \ \forall \boldsymbol{x} \neq \boldsymbol{y}$$

- There are several ways to show strict convexity, monotonicity is the best here
  - The Jansen's inequality version of strict convexity will be very tedious
  - The gradient inequality version of strict convexity will be very tedious

- Gradient: $\nabla \|\boldsymbol{x}\|_p = \dfrac{1}{\|\boldsymbol{x}\|_p^{p-1}} \Big(|x_1|^{p-1} \mathrm{sign}(x_1),\ \ldots,\ |x_n|^{p-1} \mathrm{sign}(x_n) \Big).$

- Strict convexity $\iff$

$$\sum_{i=1}^{n} \left(\frac{|x_i|^{p-1} \mathrm{sign}(x_i)}{\|\boldsymbol{x}\|_p^{p-1}} - \frac{|y_i|^{p-1} \mathrm{sign}(y_i)}{\|\boldsymbol{y}\|_p^{p-1}} \right) (x_i - y_i), \quad \forall \boldsymbol{x} \neq \boldsymbol{y}$$

Metric projection $\mathrm{proj}_{U,p}$ / best ℓ_p approximation

- **Given** $\begin{cases} \boldsymbol{x} \in \mathbb{R}^n & \text{a point} \\ U \subset \mathbb{R}^n & \text{a set} \end{cases}$

  Find $\boldsymbol{u} \in U$ by solving

$$\mathrm{proj}_{U,p}(\boldsymbol{x}) \coloneqq \underset{\boldsymbol{u} \in U}{\mathrm{argmin}} \; \|\boldsymbol{x} - \boldsymbol{u}\|_p \qquad \text{(Metric projection onto } U \text{ wrt } p\text{-norm)}$$

- Name
  - Projection
  - Metric projection
  - Best ℓ_p approximation

- The set $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is possibly
  - empty: no solution
  - singleton: solution exists, and solution is unique
  - set-valued: solution exists, and solution is not unique

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Geometry of projection $\mathrm{proj}_{U,p}$: nonempty closed convex polytope

Uniqueness of $\mathbf{proj}_{U,p}(\boldsymbol{x}) \coloneqq \underset{\boldsymbol{u}\in U}{\mathbf{argmin}}\ \|\boldsymbol{x}-\boldsymbol{u}\|_p$

$\boxed{U \subset \mathbb{R}^n}$

- For $1 < p < \infty$, the mapping $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is a $\underbrace{\text{point-valued map}}_{\text{the minimizer is unique}}$
- Fact: ℓ_p-norm with $1 < p < \infty$ is strictly convex (next page)
- Assumption: U is nonempty, closed, convex
- Then,

$$\begin{cases} 1 < p < \infty \implies \|\boldsymbol{x}-\boldsymbol{u}\|_p \text{ is strictly convex} \\ U \text{ is nonempty, closed, convex} \end{cases} \implies \underset{\boldsymbol{u}\in U}{\mathrm{argmin}}\ \|\boldsymbol{x}-\boldsymbol{u}\|_p \text{ has unique solution}$$

Uniqueness of $\mathbf{proj}_{U,p}(\boldsymbol{x}) \coloneqq \mathop{\mathbf{argmin}}_{\boldsymbol{u}\in U} \|\boldsymbol{x}-\boldsymbol{u}\|_p$

$\boxed{U \subset \mathbb{R}^n}$

- For $p \in \{1, \infty\}$, the mapping $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is generally $\underbrace{\text{set-valued map}}_{\text{the minimizer is non-unique}}$
- $\mathrm{proj}_{U,p}(\boldsymbol{x})$ may contain infinite-many elements
- Summary

$$\mathrm{proj}_{U,p}(\boldsymbol{x}) \begin{cases} \text{has unique solution} & 1 < p < \infty \\ \text{may has non-unique solution} & p \in \{1, \infty\} \end{cases}$$

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Point-to-set distance and Hausdorff distance

- $\mathrm{dist}(\boldsymbol{x}, S) := \min_{\boldsymbol{u} \in S} \|\boldsymbol{x} - \boldsymbol{u}\|$ is the distance from the point $\boldsymbol{x}$ to the set S

- $\max \mathrm{dist}(\boldsymbol{x}, S)$ is the furthest distance between the point $\boldsymbol{x}$ to the set S
  - the furthest you can reach from the set S to the point $\boldsymbol{x}$

- $\max_{\boldsymbol{x} \in X} \mathrm{dist}(\boldsymbol{x}, S)$ is the furthest distance between the point $\boldsymbol{x} \in X$ to the set S
  - the furthest you can reach from the set S to the set X

- Hausdorff distance: symmetric version of the distance

$$\mathrm{dist}_H(S_1, S_2) := \max \left\{ \max_{\boldsymbol{x} \in S_1} \mathrm{dist}(\boldsymbol{x}, S_2), \quad \max_{\boldsymbol{y} \in S_2} \mathrm{dist}(\boldsymbol{y}, S_1) \right\}$$

$\mathrm{proj}_{U,p}(x)$ is Lipschitz continuous

- For $1 < p < \infty$, the projection mapping $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is Lipschitz continuous

$$\left\|\mathrm{proj}_{U,p}(\boldsymbol{x}) - \mathrm{proj}_{U,p}(\boldsymbol{y})\right\|_2 \leq \gamma\|\boldsymbol{x} - \boldsymbol{y}\|_2 \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$$

  - that is, there exists constant $\gamma > 0$ such that the above inequality is true
  - In fact, $\gamma = 1$
  - $\gamma = 1$ means the projection map is 1-Lipschitz and nonexpansive (details)
  - $\left\|\mathrm{proj}_{U,p}(\boldsymbol{x}) - \mathrm{proj}_{U,p}(\boldsymbol{y})\right\|_2$ is the ℓ_2-norm of $\dim(U)$ vector, while $\|\boldsymbol{x} - \boldsymbol{y}\|_2$ is the ℓ_2-norm of n-vector
  - in this case $\mathrm{proj}_{U,p}$ is a Lipschitz continuous map from $\mathbb{R}^n$ to $\mathbb{R}^{\dim(U)}$

- For $p \in \{0, \infty\}$, the projection mapping $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is Lipschitz continuous in the Hausdorff metric

$$\mathrm{dist}_H\Big(\mathrm{proj}_{U,p}(\boldsymbol{x}),\ \mathrm{proj}_{U,p}(\boldsymbol{y})\Big) \leq \gamma\|\boldsymbol{x} - \boldsymbol{y}\|_2 \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$$

  - that is, there exists constant $\gamma > 0$ such that the above inequality is true
  - lack of strict convexity, now we don't know γ

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Geometry of projection: $\mathrm{proj}_{U,p}(x)$ is a nonempty closed convex polytope

If $\begin{cases} 1 \leq p \leq \infty \\ U \subset \mathbb{R}^n \text{ is a } \underbrace{\text{nonempty closed convex polytope}}_{\text{(bounded intersection of finitely many halfspaces)}} \end{cases}$

Then $\forall \boldsymbol{x} \in \mathbb{R}^n$

$$\mathrm{proj}_{U,p}(\boldsymbol{x}) := \operatorname*{argmin}_{\boldsymbol{u} \in U} \|\boldsymbol{x} - \boldsymbol{u}\|_p$$

is a nonempty closed convex polytope (possibly a singleton).

- $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is nonempty
  - $\boldsymbol{u} \mapsto \|\boldsymbol{x} - \boldsymbol{u}\|_p$ is convex for $p \geq 1$
  - $\boldsymbol{u} \mapsto \|\boldsymbol{x} - \boldsymbol{u}\|_p$ is is continuous and coercive
  - therefore projection exists ($\mathrm{proj}_{U,p}(\boldsymbol{x}) \neq \varnothing$)
  - details here

Geometry of projection: $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is a nonempty closed convex polytope

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is a nonempty closed convex polytope (possibly a singleton).

- $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is a closed set
  - $\|\boldsymbol{x} - \boldsymbol{u}\|_p$ is continuous
  - argmin of continuous function over a closed set is closed

Geometry of projection: $\operatorname{proj}_{U,p}(\boldsymbol{x})$ is a nonempty closed convex polytope

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is a nonempty closed convex polytope (possibly a singleton).

- $\operatorname{proj}_{U,p}(\boldsymbol{x})$ is a convex set
  - $\boldsymbol{u} \mapsto \|\boldsymbol{x} - \boldsymbol{u}\|_p$ is convex
  - convex function has convex sublevel set
  - $\operatorname{lev}_{\le \alpha} = \{\boldsymbol{u} \in U \ : \ \|\boldsymbol{x} - \boldsymbol{u}\|_p \le \alpha\}$ is convex for any α

Geometry of projection: $\mathrm{proj}_{U,p}(x)$ is a nonempty closed convex polytope

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is a nonempty closed convex polytope (possibly a singleton).

- $\mathrm{proj}_{U,p}(\boldsymbol{x})$ is a polytope (possibly a singleton)
  - U is a nonempty closed convex polytope
  - The minimizers of a convex function over a polytope form a closed facet of U
  - facet the polytope is a polytope

Last page - summary

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