Convergence analysis of first-order methods on quadratic cost function by discrete time dynamic system

Andersen Ang

Mathématique et recherche opérationnelle
UMONS, Belgium

manshun.ang@umons.ac.be Homepage: angms.science

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Problem set up

- Consider a unconstrained quadratic programming problem

$$(\mathcal{P}) \ : \ \min_{\mathbf{x}} f(\mathbf{x}), \text{ where } f(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}, \mathbf{Q}\mathbf{x} \rangle - \langle \mathbf{p}, \mathbf{x} \rangle + c,$$

  with $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $m\mathbf{I} \preceq \mathbf{Q} \preceq L\mathbf{I}$ and $\mathbf{p} \in \mathbb{R}^n$.

- As the problem is QP, it is a convex problem.

- For $m > 0$, the problem is strongly convex and so the minimizer $\mathbf{x}_*$ exists and is unique. The minimizer $\mathbf{x}_*$ can be determined by the first-order optimality condition (FoC): the gradient of f is

$$\nabla f(\mathbf{x}) = \mathbf{Q}\mathbf{x} - \mathbf{p},$$

  and the minimizer is $\mathbf{x}_*$ satisfies $\nabla f(\mathbf{x}_*) = 0$, which gives $\mathbf{Q}\mathbf{x}_* = \mathbf{p}$.

Fist-order Method (FoM)

- Consider we solve Problem $(\mathcal{P})$ by a First-order Method (FoM), for example the Gradient Descent (GD)

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha \nabla f(\mathbf{x}_k),$$

  where $\alpha \geq 0$ is stepsize.

- By the theory of gradient descent, we know that the sequence produced by GD converges.

- We now show the same conclusion using linear discrete time dynamic system with feedback.

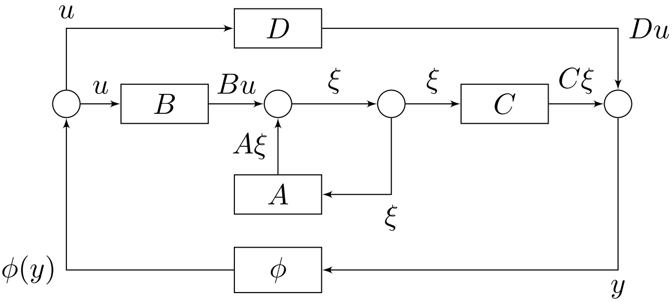
Linear discrete time dynamic system with feedback

- The state space model of linear discrete time dynamic system

$$\begin{cases} \xi_{k+1} & = & \mathbf{A}\xi_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k & = & \mathbf{C}\xi_k + \mathbf{D}\mathbf{u}_k \\ \mathbf{u}_k & = & \phi(\mathbf{y}_k) \end{cases}$$

where ξ_k is the state variable, $\mathbf{y}_k$ is the output and $\mathbf{u}_k$ is the input. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are coefficient matrices; the function ϕ is (in general) a nonlinear function.

- Block diagram



Tools for the convergence analysis using dynamic system

- To study the convergence analysis of FoM in terms of dynamic system

$$\begin{cases} \xi_{k+1} & = & \mathbf{A}\xi_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k & = & \mathbf{C}\xi_k + \mathbf{D}\mathbf{u}_k \\ \mathbf{u}_k & = & \phi(\mathbf{y}_k) \end{cases}$$

  we make use of the following
  - FoC: at optimal point, $\mathbf{u}_* = \nabla f(\mathbf{y}_*) = 0$. This gives

    $$\begin{cases} \xi_* & = & \mathbf{A}\xi_* \\ \mathbf{y}_* & = & \mathbf{C}\xi_* \end{cases},$$

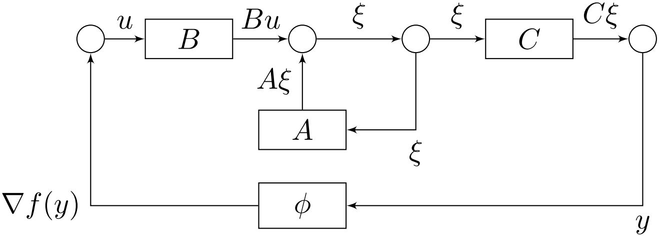
    $\xi_* = \mathbf{A}\xi_*$ implies $\|\xi_*\| = \|\mathbf{A}\xi_*\|$, by operator-norm inequality $\|\xi_*\| \leq \|\mathbf{A}\|\|\xi_*\|$, which gives $1 \leq \|\mathbf{A}\|$.
  - Let the state variable ξ_k be the hidden variable deal with the dynamics of the system.
  - Let the output variable $\mathbf{y}_k$ be the observed variable after applying the observation operator $\mathbf{C}$, while assuming $\mathbf{D} = 0$.
  - The ϕ function is the gradient function $\nabla f(\cdot)$.

Simplified dynamic system

- With $\mathbf{D} = 0$, $\phi = \nabla f(\,\cdot\,)$, the dynamic system becomes

$$\begin{cases} \xi_{k+1} & = & \mathbf{A}\xi_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k & = & \mathbf{C}\xi_k \\ \mathbf{u}_k & = & \nabla(\mathbf{y}_k) \end{cases} .$$

- Corresponding block diagram



Convergence analysis using dynamic system ... 1/3

- The general idea of the convergence analysis using dynamic system

$$\begin{cases} \xi_{k+1} & = & \mathbf{A}\xi_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k & = & \mathbf{C}\xi_k \\ \mathbf{u}_k & = & \nabla(\mathbf{y}_k) \end{cases}$$

  is to reduce the three equations into one on the state variable ξ only.

- To start, we make use of the gradient of the QP and FoC gives $\nabla(\mathbf{y}_k) = \mathbf{Q}\mathbf{y}_k - \mathbf{p} \overset{\text{FoC}}{=} \mathbf{Q}\mathbf{y}_k - \mathbf{Q}\mathbf{y}_*$. Now can eliminate the third equation and get

$$\begin{cases} \xi_{k+1} & = & \mathbf{A}\xi_k + \mathbf{B}\mathbf{Q}(\mathbf{y}_k - \mathbf{y}_*) \\ \mathbf{y}_k & = & \mathbf{C}\xi_k \end{cases}.$$

- By FoC: $\mathbf{y}_* = \mathbf{C}\xi_*$, we eliminate $\mathbf{y}$ and get

$$\xi_{k+1} = \mathbf{A}\xi_k + \mathbf{BQC}(\xi_k - \xi_*).$$

  Now we have reduced the dynamic system into just one equation on ξ, the reaming job is to bound the distance $\|\xi_{k+1} - \xi_*\|$.

Convergence analysis using dynamic system ... 2/3

- Some algebra

$$\begin{aligned} \xi_{k+1} &= \mathbf{A}\xi_k + \mathbf{BQC}(\xi_k - \xi_*) \\ &= (\mathbf{A} + \mathbf{BQC})\xi_k - \mathbf{BQC}\xi_* \end{aligned}$$

- By FoC: $\xi_* = \mathbf{A}\xi_*$,

$$\begin{aligned} \xi_{k+1} - \xi_* &= (\mathbf{A} + \mathbf{BQC})\xi_k - \mathbf{BQC}\xi_* - \xi_* \\ &= (\mathbf{A} + \mathbf{BQC})\xi_k - \mathbf{BQC}\xi_* - \mathbf{A}\xi_* \\ &= (\mathbf{A} + \mathbf{BQC})\xi_k - (\mathbf{A} + \mathbf{BQC})\xi_* \\ &= (\mathbf{A} + \mathbf{BQC})(\xi_k - \xi_*) \\ &= (\mathbf{A} + \mathbf{BQC})^2(\xi_{k-1} - \xi_*) \\ &\;\;\vdots \\ &= (\mathbf{A} + \mathbf{BQC})^k(\xi_0 - \xi_*). \end{aligned}$$

- Let $\mathbf{T} = \mathbf{A} + \mathbf{BQC}$ be the transfer matrix. Take norm and by norm inequality

$$\|\xi_{k+1} - \xi_*\| = \|\mathbf{T}^k(\xi_0 - \xi_*)\| \leq \|\mathbf{T}\|^k \|\xi_0 - \xi_*\|,$$

and the sequence converge if $\|\mathbf{T}\| < 1$.

Convergence analysis using dynamic system ... 3/3

- For GD step $\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha f(\mathbf{x}_k)$, we can take
  - $\xi_k = \mathbf{x}_k$, $\mathbf{y}_k = \xi_k$, $\mathbf{u}_k = \nabla f(\mathbf{y}_k)$
  - $\mathbf{A} = \mathbf{I}$, $\mathbf{B} = -\alpha\mathbf{I}$, $\mathbf{C} = \mathbf{I}$, $\mathbf{D} = 0$
- On the QP problem the dynamic system $\mathbf{T}$ matrix becomes $\mathbf{T} = \mathbf{I} - \alpha\mathbf{Q}$, and since $\mathbf{Q} \succeq L\mathbf{I}$, thus

  $$\|\mathbf{T}\| = \|\mathbf{I} - \alpha\mathbf{Q}\| \leq \|\mathbf{I} - \alpha L\mathbf{I}\| = \|(1-\alpha L)\mathbf{I}\| \leq |1-\alpha L| \cdot \|\mathbf{I}\| = |1-\alpha L|.$$
- Now $\|\mathbf{T}\| < 1$ if $|1 - \alpha L| < 1$. The absolute value inequality gives two cases: $1 - \alpha L < 1$ or $1 - \alpha L > -1$.
  - The first case gives $\alpha > 0$ (stepsize is positive)
  - The second case gives $\alpha < \dfrac{2}{L}$, which is what we know from the theory of gradient descent.
- That is, we have just re-confirmed that for stepsize $0 < \alpha < \dfrac{2}{L}$, the sequence produced by GD update on solving the QP problem converge.

Last page - summary

- The approach used works for QP problem.
- What about general problem that is not QP? See the paper: Lessard et al., Analysis and Design of Optimization Algorithms via Integral Quadratic Constraints.

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