

Characterizing subdifferential of norm

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$$

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The statement

- ▶ Let $\|\cdot\|$ be a vector norm in \mathbb{R}^n . Let $\partial(\cdot)$ be subdifferential, then

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}, \quad (1)$$

where

$$\|x\|_* := \sup_{\|u\| \leq 1} \langle x, u \rangle$$

is the dual norm of $\|\cdot\|$.

- ▶ What (1) means: the subdifferential of norm at a point x , is the set of vector v as described in (1), and such set characterizes all the possible descent direction of the norm function.
- ▶ This document: prove (1).

Proof by equivalence of sets

- ▶ Recall: subdifferential of a function f at a point x is the set of subgradients of f at x .
- ▶ To prove (1), what we do is to show the set of subgradients equals to the following set

$$V(x) = \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\},$$

by showing :

- ▶ $V(x)$ is a subset of $\partial\|x\|$
If $v \in \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$, then $v \in \partial\|x\|$.
- ▶ $\partial\|x\|$ is a subset of $V(x)$
If $v \in \partial\|x\|$, then $v \in \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$.
- ▶ Then, $V(x) \subset \partial\|x\|$ and $\partial\|x\| \subset V(x)$ implies $\partial\|x\| = V(x)$, which proves (1).

Pre-requisite

► Subgradient

For a function f , v is a subgradient of f at x if and only if

$$f(y) \geq f(x) + \langle v, y - x \rangle \quad \forall y.$$

► Holder's inequality

For any dual pair of norms, we have

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|_*.$$

► Fenchel conjugate of norm is indicator function on unit ball of dual norm

$$\|x\|^* = \sup_u \langle x, u \rangle - \|u\| = i_{\|x\|_* \leq 1}(x) = \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

See the proof here.

The proof : $V(x) \subset \partial\|x\|$

- ▶ Let $v \in V(x) = \left\{ v \in \mathbb{R}^n \mid \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$.
- ▶ Let y be arbitrary vector. Then

$$\begin{aligned} \|x\| + \langle v, y - x \rangle &= \|x\| + \langle v, y \rangle - \langle v, x \rangle \\ &= \|x\| + \langle v, y \rangle - \|x\| && \because v \in V(x) \\ &= \langle v, y \rangle \\ &\leq \|y\| \cdot \|v\|_* && \text{Holder's inequality} \\ &\leq \|y\| && \because v \in V(x) \end{aligned}$$

- ▶ As y is arbitrary, so we established the following inequality for all y :

$$\|y\| \geq \|x\| + \langle v, y - x \rangle,$$

which is the definition of subgradient of norm function, so $v \in \partial\|x\|$.
This is true for all $v \in \partial\|x\|$ and thus $V(x) \subset \partial\|x\|$.

The proof : $\partial\|x\| \subset V(x) \dots 1/2$

- ▶ Let $v \in \partial\|x\|$. By definition of subgradient,

$$\|y\| \geq \|x\| + \langle v, y - x \rangle, \quad \forall y.$$

Rearrange gives

$$\langle v, y \rangle - \|y\| \leq \langle v, x \rangle - \|x\|, \quad \forall y$$

- ▶ The inequality holds for all y , taking supremum on y does not change the inequality

$$\underbrace{\sup_y \langle v, y \rangle - \|y\|}_{\|v\|^*} \leq \langle v, x \rangle - \|x\|, \quad \forall y$$

where $\|v\|^* = \sup_y \langle v, y \rangle - \|y\|$ is the Fenchel conjugate of norm at the point v , which is the indicator function on the unit ball of dual norm. So we have

$$\langle v, x \rangle - \|x\| \geq \begin{cases} 0 & \|v\|_* \leq 1 \\ +\infty & \|v\|_* > 1 \end{cases}.$$

The proof : $\partial\|x\| \subset V(x) \dots 2/2$

- ▶ The case $\|v\|_* > 1$ is impossible as $\langle v, x \rangle - \|x\|$ is always finite.
Hence we have

$$\langle v, x \rangle - \|x\| \geq 0 \iff \|v\|_* \leq 1.$$

- ▶ As $\|v\|_* \leq 1$, then the inequality becomes

$$\begin{aligned} 0 &\leq \langle v, x \rangle - \|x\| \\ &\leq \|x\| \cdot \|v\|_* - \|x\| && \text{Holder's inequality} \\ &= \|x\|(\|v\|_* - 1) \\ &\leq 0 && \because \|v\|_* \leq 1 \end{aligned}$$

So we have $0 \leq \langle v, x \rangle - \|x\| \leq 0$, that is: $\langle v, x \rangle = \|x\|$.

- ▶ So we showed $v \in V(x)$ and thus $\partial\|x\| \subset V(x)$.
The whole proof is completed. □

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