Characterizing subdifferential of norm

Andersen Ang

ECS, Uni. Southampton, UK

andersen.ang@soton.ac.uk

Homepage angms.science

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$$

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The statement

- Let $\|\cdot\|$ be a vector norm in $\mathbb{R}^n$ and let ∂ denotes subdifferential, then

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x\rangle = \|x\|,\ \|v\|_* \leq 1 \right\}, \tag{1}$$

  where $\|x\|_*$ is the dual norm of $\|\cdot\|$, defined as

$$\|x\|_* := \sup_{\|u\| \leq 1} \langle x, u\rangle.$$

- (1) means: the subdifferential of norm at a point x, is a set of vector v described in (1).
- This document: prove (1).

Proof by equivalence of sets

- Recall: subdifferential of a function f at a point x is the *set* of subgradients of f at x.

- How to prove (1): show the set of subgradients equals to the following set

$$V(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \le 1 \right\},$$

  by showing :

  - $V(x)$ is a subset of $\partial\|x\|$ $\quad v \in \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \le 1 \right\}$ implies $v \in \partial\|x\|$.
  - $\partial\|x\|$ is a subset of $V(x)$ $\quad v \in \partial\|x\|$ implies $v \in \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \le 1 \right\}$.

- Then $\begin{cases} V(x) \subset \partial\|x\| \\ \partial\|x\| \subset V(x) \end{cases}$ imply $\partial\|x\| = V(x)$, which proves (1).

Pre-requisite

- **Subgradient**: For a function f, v is a subgradient of f at x if and only if

$$f(y) \geq f(x) + \langle v, y - x \rangle \ \ \forall\, y.$$

- **Holder's inequality** For any dual pair of norms, we have

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|_*.$$

- **Fenchel conjugate of norm is indicator function on unit ball of dual norm**

$$\|x\|^* = \sup_u \langle x, u \rangle - \|u\| \ = \ i_{\|x\|_* \leq 1}(x) = \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

See the proof here.

The proof : $V(x) \subset \partial\|x\|$

- Let $v \in V(x) = \left\{ v \in \mathbb{R}^n \,\middle|\, \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$.
- Let y be an arbitrary vector. Then

$$\begin{array}{rcll} \|x\| + \langle v, y - x \rangle & = & \|x\| + \langle v, y \rangle - \langle v, x \rangle & \\ & = & \|x\| + \langle v, y \rangle - \|x\| & \because v \in V(x) \\ & = & \langle v, y \rangle & \\ & \leq & \|y\| \cdot \|v\|_* & \text{Holder's inequality} \\ & \leq & \|y\| & \because v \in V(x) \end{array}$$

- As y is arbitrary, so we established the following inequality for all y :

$$\|y\| \geq \|x\| + \langle v, y - x \rangle,$$

which is the definition of subgradient of norm function, so $v \in \partial\|x\|$. This is true for all $v \in \partial\|x\|$ and thus $V(x) \subset \partial\|x\|$.

The proof : $\partial\|x\| \subset V(x)$ ... 1/2

- Let $v \in \partial\|x\|$. By definition of subgradient,

$$\|y\| \;\geq\; \|x\| + \langle v, y - x\rangle,\ \forall\ y.$$

  Rearrange gives

$$\langle v, y\rangle - \|y\| \;\leq\; \langle v, x\rangle - \|x\|,\ \forall\ y$$

- The inequality holds for all y, taking supremum on y does not change the inequality

$$\underbrace{\sup_y \langle v, y\rangle - \|y\|}_{=:\|v\|^*} \;\leq\; \langle v, x\rangle - \|x\|,\ \forall\ y$$

  where $\|v\|^* = \sup_y \langle v, y\rangle - \|y\|$ is the Fenchel conjugate of norm at the point v, which is the indicator function on the unit ball of dual norm. So we have

$$\langle v, x\rangle - \|x\| \;\geq\; \begin{cases} 0 & \|v\|_* \leq 1 \\ +\infty & \|v\|_* > 1 \end{cases}.$$

The proof : $\partial\|x\| \subset V(x)$ ... 2/2

- The case $\|v\|_* > 1$ is impossible as $\langle v, x\rangle - \|x\|$ is always finite. Hence we have

$$\langle v, x\rangle - \|x\| \geq 0 \iff \|v\|_* \leq 1.$$

- As $\|v\|_* \leq 1$, then the inequality becomes

$$\begin{array}{rcll} 0 & \leq & \langle v, x\rangle - \|x\| & \\ & \leq & \|x\| \cdot \|v\|_* - \|x\| & \text{Holder's inequality} \\ & = & \|x\|(\|v\|_* - 1) & \\ & \leq & 0 & \because \|v\|_* \leq 1 \end{array}$$

  So we have $0 \leq \langle v, x\rangle - \|x\| \leq 0$, that is: $\langle v, x\rangle = \|x\|$.

- So we showed $v \in V(x)$ and thus $\partial\|x\| \subset V(x)$. The whole proof is completed. $\square$

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