Characterizing subdifferential of norm

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$$

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The statement

Let $\|\cdot\|$ be a vector norm in $\mathbb{R}^n$. Let $\partial(\cdot)$ be subdifferential of function. This document : show that

$$\partial\|x\| = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x\rangle = \|x\|, \|v\|_* \leq 1 \right\}, \tag{1}$$

where $\|x\|_* := \sup\limits_{\|u\|\leq 1} \langle x, u\rangle$ is the dual norm of $\|\cdot\|$.

What the statement means : the subdifferential of norm function at a point x, is the set of vector v described by (1), such set characterizes all the possible descent direction of the norm function.

This document : prove (1).

Proof by equivalence of sets

Recall that subdifferential of a function f at a point x is the *set* of subgradient of f at x.

So to prove the statement, what we do is to show the set of subgradients equals to the following set

$$V(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\},$$

by showing :

- $V(x)$ is a subset of $\partial\|x\|$
  If $v \in \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$, then $v \in \partial\|x\|$
- $\partial\|x\|$ is a subset of $V(x)$
  If $v \in \partial\|x\|$, then $v \in \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$

Then, $V(x) \subset \partial\|x\|$ and $\partial\|x\| \subset V(x)$ implies $\partial\|x\| = V(x)$, which proves the statement.

Pre-requisite

Subgradient
For a function f, v is a subgradient of f at x if and only if

$$f(y) \geq f(x) + \langle v, y - x \rangle \ \forall \ y.$$

Holder's inequality
For any dual pair of norms, we have

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|_*.$$

Fenchel conjugate of norm is indicator function on unit ball of dual norm

$$\|x\|^* = \sup_u \langle x, u \rangle - \|u\| \ = \ i_{\|x\|_* \leq 1}(x) = \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

See the proof here.

The proof : $V(x) \subset \partial\|x\|$

Let $v \in V(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle v, x \rangle = \|x\|, \|v\|_* \leq 1 \right\}$.

Let y be arbitrary vector. Then

$$\begin{aligned} \|x\| + \langle v, y - x \rangle &= \|x\| + \langle v, y \rangle - \langle v, x \rangle \\ &= \|x\| + \langle v, y \rangle - \|x\| && \because v \in V(x) \\ &= \langle v, y \rangle \\ &\leq \|y\| \cdot \|v\|_* && \text{Holder's inequality} \\ &\leq \|y\| && \because v \in V(x) \end{aligned}$$

As y is arbitrary, so we established the following inequality for all y :

$$\|y\| \geq \|x\| + \langle v, y - x \rangle,$$

which is the definition of subgradient of norm function $\| \cdot \|$.

Hence, we showed $v \in \partial\|x\|$ and thus $V(x) \subset \partial\|x\|$.

The proof : $\partial\|x\| \subset V(x)$ ... 1/2

Let $v \in \partial\|x\|$. Then by definition of subgradient, we have

$$\|y\| \geq \|x\| + \langle v, y - x\rangle, \ \forall\ y$$

Rearrange we have

$$\langle v, y\rangle - \|y\| \leq \langle v, x\rangle - \|x\|, \ \forall\ y$$

As the inequality holds for all y, taking supremum on y does not change the inequality

$$\underbrace{\sup_y \langle v, y\rangle - \|y\|}_{\|v\|^*} \leq \langle v, x\rangle - \|x\|, \ \forall\ y$$

where $\|v\|^* = \sup_y \langle v, y\rangle - \|y\|$ is the Fenchel conjugate of norm at the point v, which is the indicator function on the unit ball of dual norm. So we have

$$\langle v, x\rangle - \|x\| \geq \begin{cases} 0 & \|v\|_* \leq 1 \\ +\infty & \|v\|_* > 1 \end{cases}.$$

The proof : $\partial\|x\| \subset V(x)$ ... 2/2

The case $\|v\|_* > 1$ is impossible as $\langle v, x\rangle - \|x\|$ is always finite. Hence we have

$$\langle v, x\rangle - \|x\| \geq 0 \iff \|v\|_* \leq 1.$$

As $\|v\|_* \leq 1$, then the inequality becomes

$$\begin{array}{rcll} 0 & \leq & \langle v, x\rangle - \|x\| & \\ & \leq & \|x\| \cdot \|v\|_* - \|x\| & \text{Holder's inequality} \\ & = & \|x\|(\|v\|_* - 1) & \\ & \leq & 0 & \because \|v\|_* \leq 1 \end{array}$$

So we have $0 \leq \langle v, x\rangle - \|x\| \leq 0$, that is : $\langle v, x\rangle = \|x\|$.

So we have showed $v \in V(x)$ and thus $\partial\|x\| \subset V(x)$.
The whole proof is completed. $\square$

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