Coerce function, and

- Strongly convex function implies coerciveness
- Coerciveness implies existence of global minimum
- Weierstrass's theorem
- Euclidean projection problem always has a unique solution.

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Definition of coercive function in optimization

- A continuous function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is coercive if

$$\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = \infty.$$

  Meaning : f goes very big if $\mathbf{x}$ grows.

- As $\mathbf{x}$ grows larger in size, it can "walk pass" any values, that's why f has to be *continuous* in the definition.

- As we have ∞ as the output in f, the image of f is $\bar{\mathbb{R}}$ instead of $\mathbb{R}$. The set $\bar{\mathbb{R}}$ is the extended real line

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}.$$

- An $\epsilon - \delta$ explanation: for any constant $M > 0$, no matter how large is M, there always exists a constant R_M (radius corresponds to M) such that $\|f(\mathbf{x})\| > M$ whenever $\|\mathbf{x}\| > R_M$ (when $\mathbf{x}$ has size larger than the norm-ball with radius R_M).

f has to approach $+\infty$ in all directions to be coercive

- In the definition $\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = \infty$, there is no limitation on "how $\mathbf{x}$ should grow in size". This means that f has to approach $+\infty$ in all possible ways $\mathbf{x}$ grows. Geometrically, f has to approach $+\infty$ for any direction $\mathbf{x}$ moves.

- For example $f(\mathbf{x}) = x_1^2 + x_2^2$ is coercive since

$$\lim_{x_1 \to \infty} f(\mathbf{x}) = \infty, \quad \lim_{x_2 \to \infty} f(\mathbf{x}) = \infty, \quad \lim_{(x_1, x_2) \to \infty} f(\mathbf{x}) = \infty.$$

- $f(\mathbf{x}) = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$ is not coercive:

$$\lim_{x_1 \to \infty} f(\mathbf{x}) = \infty, \quad \lim_{x_2 \to \infty} f(\mathbf{x}) = \infty, \quad \lim_{(x_1, x_2) \to \infty} f(\mathbf{x}) \neq \infty.$$

Super-coercive

- A continuous function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is super-coercive if

$$\lim_{\|\mathbf{x}\| \to \infty} \frac{f(\mathbf{x})}{\|\mathbf{x}\|} = \infty.$$

- Super-coercive implies coercive.
- If f is strongly convex,i.e.,

$$f(\mathbf{x}) - \frac{\alpha}{2}\|\mathbf{x}\| \text{ is convex}$$

  or equivalently

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\alpha}{2}\|\mathbf{y} - \mathbf{x}\|^2$$

  then f is super-coercive[1].

[1] For proof, see Corollary 11.17 in "Convex Analysis and Monotone Operator Theory in Hilbert Spaces" by Bauschke and Combettes.

What's so special about coercive function

- **Theorem** If f is coercive, f must has a global minimum.
- *Proof* As f is coercive, meaning that the value of f goes large when $\mathbf{x}$ moves away from the point $\mathbf{y} = 0$, there for there exists a radius $R_{\mathbf{y}}$ such that for all $\mathbf{x}$ located outside the ball with such radius, the function value at $\mathbf{x}$ will be larger than or equal to $f(\mathbf{y})$. Put this mathematically,
$$\exists R > 0 \text{ s.t. } f(\mathbf{x}) \geq f(\mathbf{y}),\ \forall \|\mathbf{x}\| \geq R_{\mathbf{y}}. \quad (1)$$
By Weierstrass extreme value theorem[2], there is a global minimum $\mathbf{x}^*$ on the closed ball B centered at $\mathbf{y}$ with radius $R_{\mathbf{y}}$. Mathematically, $\exists \mathbf{x}* \in \bar{B}(\mathbf{y}, R_{\mathbf{y}})$. Note that $\mathbf{x}^*$ is a minimum, by definition of minimum, we have $f(\mathbf{x}^*) \leq f(\mathbf{y})$ and thus
$f(\mathbf{x}^*) \overset{(1)}{\leq} f(\mathbf{x}),\ \forall \|\mathbf{x}\| \geq R_{\mathbf{y}}$. So $\mathbf{x}^*$ is a global minimum. □
- Note that the point $\mathbf{y}$ is a free variable, $R_{\mathbf{y}}$ is also a free variable associated with $\mathbf{y}$, so $R_{\mathbf{y}}$ can be arbitrarily small (or large).

[2] A continuous function f on a closed bounded domain has a global minimum and maximum.

Application of coerciveness: Weierstrass's theorem

- **Theorem** For $f : \mathbb{R}^n \to (-\infty, \infty]$
  - f is closed, proper function.
  - $\mathrm{dom}\ f$ is compact.
  - f is coercive.
  - There exists a scalar γ such that the γ-level set $\left\{\mathbf{x} \mid f(\mathbf{x}) \leq \gamma\right\}$ is nonempty and compact.

  Then the set of minima of f, denoted as $\mathcal{X}^*$, is nonempty and compact.

- Proof: let $\{\gamma_k\}$ be a decreasing sequence $\lim_{k\to\infty} \gamma = f_* := \inf_{\mathbf{x}} f(\mathbf{x})$, then the set $\mathcal{X}^*$ is the intersection of all γ_k-level set. Mathematically

$$\mathcal{X}^* = \bigcap_{k=0}^{\infty} \left\{\mathbf{x} \mid f(\mathbf{x}) \leq \gamma_k\right\}.$$

  By assumption, each $\left\{\mathbf{x} \mid f(\mathbf{x}) \leq \gamma_k\right\}$ is nonempty and compact, so $\mathcal{X}^*$ is the intersection of a nested sequence of nonempty and compact sets. Hence $\mathcal{X}^*$ is nonempty and compact.

Application of coerciveness: projection exists

- **Theorem** For all $\mathbf{x} \in \mathbb{R}^n$, there exists a unique vector $P_C(\mathbf{x})$ that minimizes $\|\mathbf{z} - \mathbf{x}\|$ over all $\mathbf{z} \in C$.

- Proof: fix $\mathbf{x}$ and let $\mathbf{w}$ be some element of C. Minimizing $\|\mathbf{x} - \mathbf{z}\|$ over all $\mathbf{z} \in C$ is equivalent to minimizing the continuous function $g(\mathbf{z}) = \|\mathbf{z} - \mathbf{x}\|^2$ over the set of all $\mathbf{z} \in C$ such that $\|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{w}\|$, which is a compact set. Hence there exists a minimizing vector by Weierstrass, which is unique since $\|\cdot\|^2$ is strictly convex.

- Special case: if $\|\cdot\|$ is the Euclidean norm, the following problem always has a unique minimizer

$$\operatorname*{argmin}_{\mathbf{z} \in C} \frac{1}{2}\|\mathbf{z} - \mathbf{x}\|_2^2,$$

  where C is nonempty and compact set.

- Proof: $\|\mathbf{z} - \mathbf{x}\|_2^2$ is 1-strongly convex, so it is coercive.

Last page - summary

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