Fenchel conjugate of norm
$=$ indicator function on unit ball of dual norm

Andersen Ang

Mathématique et recherche opérationnelle, UMONS, Belgium

manshun.ang@umons.ac.be   Homepage: angms.science

First draft: Janurary 3, 2019
Last update: December 22, 2020

- **Definition (Fenchel conjugate)** The Fenchel conjugate of a function f at a point $\mathbf{x}$, denoted as $f^*(x)$, is defined as

$$f^*(x) = \sup_{u \in \text{dom } f} \langle x, u \rangle - f(u),$$

where u here is a dummy variable.

- **Definition (Dual norm)** The dual norm of a given $\|\cdot\|$ at the point x, denoted as $\|x\|_*$, is defined as

$$\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u \rangle,$$

again u here is a dummy variable.

- **Definition (Indicator function)** The indicator function of a given set C at a point x, denoted as $i_C(x)$, is defined as

$$i_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases}$$

It gives no cost if $x \in C$, and gives infinite cost if x is outside C.

Conjugate of norm

- The statement: "Fenchel conjugate of norm is the indicator function on unit ball of dual norm" is:

$$\|x\|^* = \sup_u \langle x, u\rangle - \|u\| = i_{\|x\|_* \le 1}(x) = \begin{cases} 0 & \|x\|_* \le 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

  where:

$$\begin{array}{rcll} f(x)^* & = & \sup_u \langle x, u\rangle - f(u) & \text{Definition of conjugate} \\ \|x\|^* & = & \sup_u \langle x, u\rangle - \|u\| & \text{Put } f(\cdot) \text{ as norm} \\ i_C(x) & = & \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} & \text{Definition of indicator function} \\ i_{\|x\|_* \le 1}(x) & = & \begin{cases} 0 & \|x\|_* \le 1 \\ +\infty & \|x\|_* > 1 \end{cases} & \text{Replace } C \text{ by the unit ball of } \|x\|_* \end{array}$$

- This document: prove this by showing the equality in green.

Framework of the proof

- We show

$$\|x\|^* = \sup_u \langle x, u\rangle - \|u\| = i_{\|x\|_* \leq 1}(x) = \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

  using direct proof by considering two cases :

- For $\|x\|_* \leq 1$, show $\|x\|^* = 0$.
- For $\|x\|_* > 1$, show $\|x\|^* = \infty$.
- The key: use the definition of dual norm

$$\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u\rangle.$$

The proof ... case 1: $\|x\|_* \leq 1$

- By the definition of dual norm $\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u \rangle \leq 1$.
- Then, we have
$$\langle x, u \rangle \leq \|x\|_* \|u\| \ \forall \ u$$
Why: if $u = 0$, it is trivial. If $u \neq 0$, then $\dfrac{u}{\|u\|} = 1 \leq 1$ so
$$\left\langle x, \frac{u}{\|u\|} \right\rangle \leq \sup_{\|u\| \leq 1} \langle x, u \rangle = \|x\|_*. \tag{1}$$
- Multiply (1) with $\|u\|$ gives
$$\left\langle x, \frac{u}{\|u\|} \right\rangle \|u\| = \langle x, u \rangle \leq \sup_{\|u\| \leq 1} \langle x, u \rangle \|u\| \overset{(1)}{=} \|x\|_* \|u\|.$$
- The blue inequality
$$\langle x, u \rangle \leq \|x\|_* \|u\| \quad \Longrightarrow \quad \sup_u \langle x, u \rangle \leq \|x\|_* \|u\| \ \forall \ u.$$
- As $\|x\|_* \leq 1$, so
$$\sup_u \langle x, u \rangle \leq \|u\| \ \forall \ u, \tag{2}$$
where equality hold if $u = 0$.
- Hence $\sup_u \langle x, u \rangle - \|u\| \overset{(2)}{=} \|u\| - \|u\| = 0$. We showed $\|x\|_* \leq 1$ implies $\|x\|^* = 0$.

The proof ... case 2 $\|x\|_* > 1$

- By the definition of dual norm $\|x\|_* = \sup_{\|u\|\leq 1}\langle x, u\rangle > 1$. So there exists a u such that

$$\|u\| \leq 1, \langle x, u\rangle > 1. \tag{3}$$

- By the definition of conjugate

$$\|x\|^* = \sup_u \langle x, u\rangle - \|u\|. \tag{4}$$

- Put (3) into (4) gives $\langle x, u\rangle - \|u\| > 0$. Then we show in this case $\|x\|^* \overset{(4)}{=} \sup_u \langle x, u\rangle - \|u\| = \infty$ by considering $u = tz$

$$\langle x, u\rangle - \|u\| = t(\langle x, z\rangle - \|z\|), \tag{5}$$

  which is unbounded (can let $\lim t \to \infty$), hence the sup of (5) $= \infty$.

- So we showed $\|x\|_* > 1$ implies $\|x\|^* = \infty$. The proof is finished. $\square$

End of document