# Fenchel conjugate of norm is indicator function on unit ball of dual norm

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# Definitions

**Definition (Fenchel conjugate)** Given a function $f(\cdot)$, the Fenchel conjugate of $f$ at the point $x$, denoted as $f^*(x)$, is defined as

$$f^*(x) = \sup_{u \in \textsf{dom } f} \langle x, u \rangle - f(u)$$

**Definition (Dual norm)** Given a norm $\|\cdot\|$, the dual norm of $\|\cdot\|$ at the point $x$, denoted as $\|x\|_*$, is defined as

$$\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u \rangle$$

**Definition (Indicator function)** Given a set $C$, the indicator function of $C$ at the point $x$, denoted as $i_C(x)$, is defined as

$$i_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases}$$

The variable $u$ here is a dummy variable.

# Conjugate of norm

The statement : "fenchel conjugate of norm is indicator function on unit ball of dual norm" is therefore :

$$\|x\|^* = \sup_u \langle x, u \rangle - \|u\| = i_{\|x\|_* \leq 1}(x) = \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

where :

$$\begin{array}{rcll} f(x)^* & = & \sup_u \langle x, u \rangle - f(u) & \text{Definition of conjugate} \\ \|x\|^* & = & \sup_u \langle x, u \rangle - \|u\| & \text{Put } f(\cdot) \text{ as norm} \\ i_C(x) & = & \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} & \text{Definition of indicator function} \\ i_{\|x\|_* \leq 1}(x) & = & \begin{cases} 0 & \|x\|_* \leq 1 \\ +\infty & \|x\|_* > 1 \end{cases} & \text{Replace } C \text{ by the unit ball of } \|x\|_* \end{array}$$

This document : prove this by showing the equality in blue.

# The framework of the proof

We show

$$\|x\|^* = \sup_u \langle x, u\rangle - \|u\| = i_{\|x\|_* \le 1}(x) = \begin{cases} 0 & \|x\|_* \le 1 \\ +\infty & \|x\|_* > 1 \end{cases}.$$

by direct proof by considering the two cases :

- For $\|x\|_* \le 1$, show $\|x\|^* = 0$
- For $\|x\|_* > 1$, show $\|x\|^* = \infty$

The key : use the definition of dual norm :

$$\|x\|_* = \sup_{\|u\| \le 1} \langle x, u\rangle$$

# The proof ... case 1

Consider $\|x\|_* \leq 1$. By the definition of dual norm, we have

$$\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u \rangle \leq 1.$$

Then, we have

$$\langle x, u \rangle \leq \|x\|_* \|u\| \ \forall \ u$$

why : if $u = 0$, it is trivial.
If $u \neq 0$, then we have $\dfrac{u}{\|u\|} = 1 \leq 1$ so $\langle x, \dfrac{u}{\|u\|} \rangle \leq \sup\limits_{\|u\| \leq 1} \langle x, u \rangle = \|x\|_*$.
Multiply with $\|u\|$ we have $\langle x, u \rangle \leq \sup\limits_{\|u\| \leq 1} \langle x, u \rangle \|u\| = \|x\|_* \|u\|$.

As $\|x\|_* \leq 1$, therefore we have

$$\sup_u \langle x, u \rangle \leq \|u\| \ \forall \ u,$$

where equality hold if $u = 0$. Hence $\sup\limits_u \langle x, u \rangle - \|u\| = 0$.

So we showed if $\|x\|_* \leq 1$, then $\|x\|^* = 0$.

# The proof ... case 2

Now $\|x\|_* > 1$. By the definition of dual norm, we have

$$\|x\|_* = \sup_{\|u\| \leq 1} \langle x, u \rangle > 1.$$

So there exists a $u$ such that

$$\|u\| \leq 1, \langle x, u \rangle > 1. \tag{1}$$

By the definition of conjugate, we have

$$\|x\|^* = \sup_u \langle x, u \rangle - \|u\| \tag{2}$$

Put condition of (1) into (2) we have $\langle x, u \rangle - \|u\| > 0$.
Then we show in this case $\|x\|^* = \sup_u \langle x, u \rangle - \|u\| = \infty$ by considering $u = tz$

$$\langle x, u \rangle - \|u\| = t(\langle x, z \rangle - \|z\|), \tag{3}$$

which is unbounded (can let $\lim t \to \infty$), hence the sup of (3) $= \infty$.

So we showed if $\|x\|_* > 1$, then $\|x\|^* = \infty$, and finished the proof. $\square$

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