What is conjugate function?

The dual way to describe function

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First draft: November 11, 2022 Last update: November 14, 2022

Fenchel conjugate

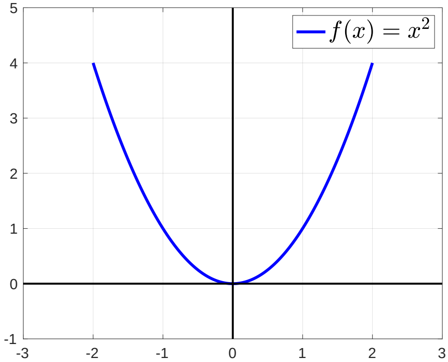
- Given a function $f(x)$, the Fenchel conjugate of f, denote as $f^*(y)$ is defined as

$$f^*(y) \ := \ \max_x \Big\{ \langle\, y,\, x \,\rangle - f(x) \Big\}.$$

- Other names of conjugate: Legendre-Fenchel transform
- So WTF is this thing?

How do we describe a function?

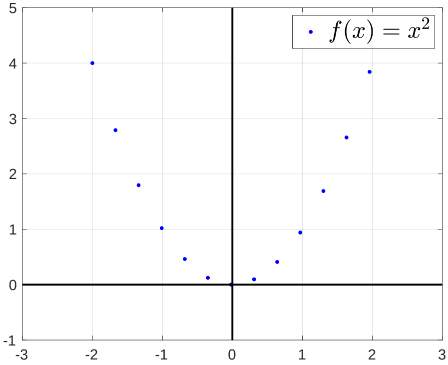
- Consider the simple function $f(x) = x^2$



- How do we "plot" a function?

We plot a function by plotting the points

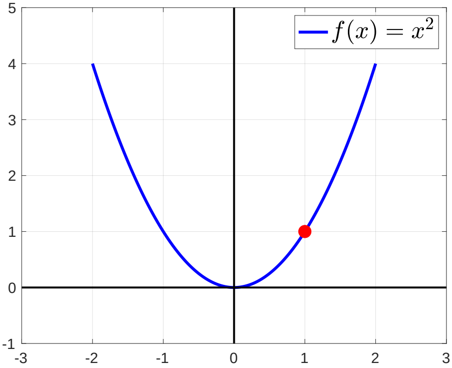
- The curve of $f(x)$ is plotted by joining a bunch of dots that lie on the function $f(x)$.
- For example



- Question: is this the only way to plot a function?
  Answer: no.

Focus on a point

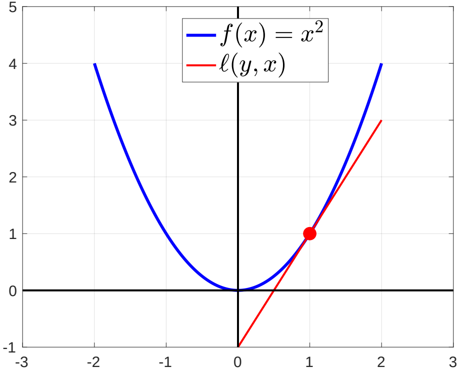
- Consider $x = 1$ (and $f(x) = 1^2 = 1$)



- The red point tells the following information
  - The x-coordinate
  - The corresponding y-coordinate

Tangent line

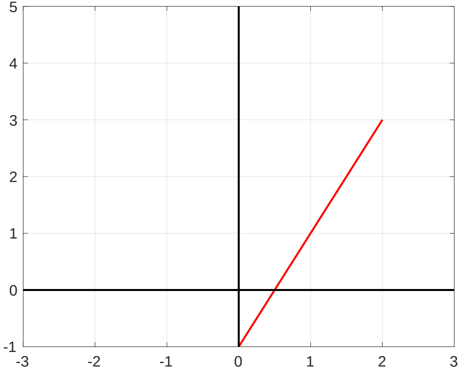
- The red point, beside tell the information of $(x, f(x))$, also tells a piece of information: the tangent line.



- The tangent line is a linear function $\ell(y, x)$
  - ℓ touch f at x
  - ℓ has the same slope of f at x
  - $\ell(y; x) = f(x) + f'(x)(y - x)$

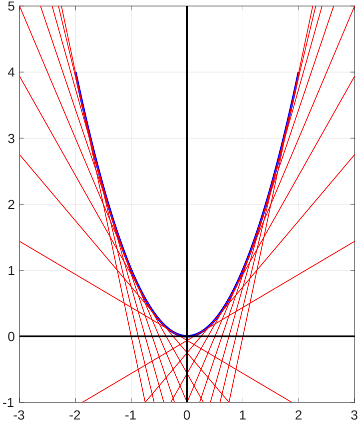
Tangent line

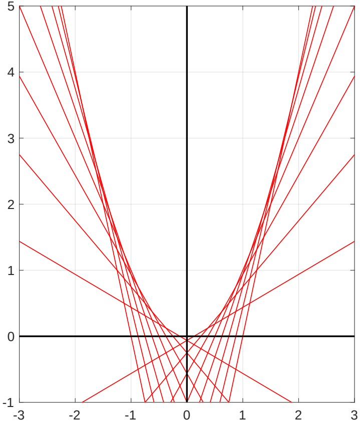
- So now, instead of having a point information $(x, f(x))$, we have a line information ℓ



Describing a function by its tangents

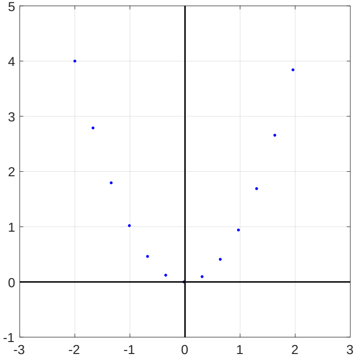
- Drawing multiple tangents also shape the function

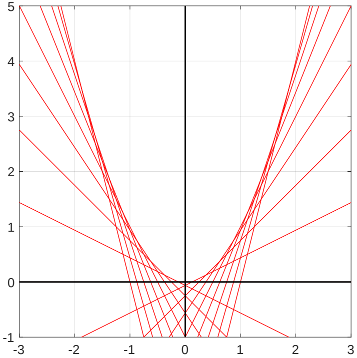




Two ways to describe a function

- We now have two ways to describe a function





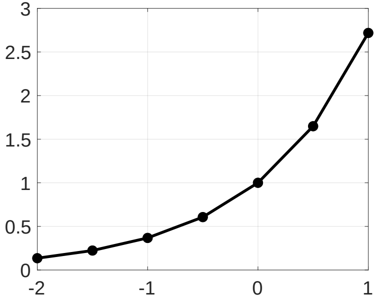
- This is a primal-dual relationship
  - Primal description: $(x, f(x))$
  - Dual description: $(\text{cross point}, \text{slope})$
- The dual description is exactly the conjugate!

Can you tell what is this function?

- Suppose I give you a series of $(x, f(x))$ as

$$(-2, 0.135),\ (-1.5, 0.223),\ (-1, 0.367),\ (-0.5, 0.606),\ (0, 1),\ (0.5, 1.648),\ (1, 2.718),$$

- You can plot these points and connect them to get



- You probably will guess this function is $f(x) = e^x$.

Can you tell what is this function?

- Suppose I give you a series of $(\xi(x), f'(x))$ as

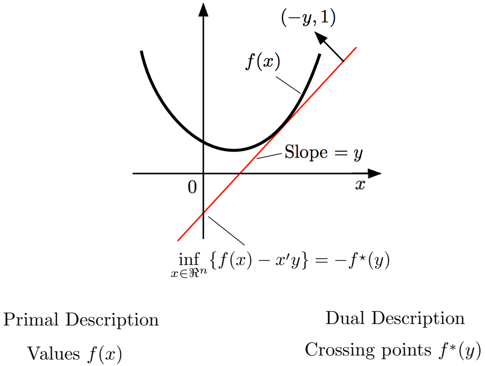
  $$(-0.135, 0.135),\ (-0.111, 0.223),\ (0, 0.367),\ (0.303, 0.606),\ (1, 1),\ (2.473, 1.648),\ (5.436, 2.718)$$

  where for a x value, $f'(x)$ is the slope at x and $\xi(x)$ is the y-axis value for the straight line with slope $f'(x)$ crossing the y-axis.

- Can you plot the straight lines and construct the function as $f(x) = e^x$?

Last slide: what is conjugate (Bertsekas's explanations)

- Define a closed convex function by its epigraph.
- Describe the epigraph by supporting hyperplanes.
- Conjugate function = crossing point of the hyperplanes



What's next: the calculus and properties of conjugate.

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