$\mathbb{G}(\mathbb{R}^2)$: Clifford Algebra of $\mathbb{R}^2$

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Standard basis in $\mathbb{R}^2$

- Vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$
  - Both unit length: $\|\boldsymbol{e}_1\| = 1$ and $\|\boldsymbol{e}_2\| = 1$.
  - Orthogonal: $\boldsymbol{e}_1 \perp \boldsymbol{e}_2$
- All vector $\boldsymbol{v} \in \mathbb{R}^2$ is a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$

$$\boldsymbol{v} = \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2,$$

where $\alpha, \beta \in \mathbb{R}$ are *coefficients*, or the *coordinate* of $\boldsymbol{v}$ in the basis $\boldsymbol{e}_1, \boldsymbol{e}_2$.

Dot product $\boldsymbol{u} \cdot \boldsymbol{v}$

- A vector-to-scalar operation
  - Input: two vectors $\boldsymbol{u}, \boldsymbol{v}$
  - Output: a scalar

- Definition: suppose $\boldsymbol{u} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ and $\boldsymbol{v} = c\boldsymbol{e}_1 + d\boldsymbol{e}_2$. The dot product between $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted as $\boldsymbol{u} \cdot \boldsymbol{v}$, is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = ac + bd.$$

- Important aspects of dot product
  - Projection: it tells how much a vector is in the same direction to another vector.
  - Orthogonality: if $\boldsymbol{u} \perp \boldsymbol{v}$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

- Property of dot product
  - Commutative: $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.
  - $\boldsymbol{u} \cdot \boldsymbol{u} = \|\boldsymbol{u}\|^2$.

Wedge product $\boldsymbol{u} \wedge \boldsymbol{v}$

- A vector-to-bivector operation
  - Input: two vectors $\boldsymbol{u}, \boldsymbol{v}$
  - Output: a bivector

- $\boldsymbol{u} \wedge \boldsymbol{v}$ is called a bivector.

- What it is: the parallelogram where $\boldsymbol{u}, \boldsymbol{v}$ serve as the two sides of it.

- The magnitude of $\boldsymbol{u} \wedge \boldsymbol{v}$ is the area of such parallelogram.
  $\implies \boldsymbol{u} \wedge \boldsymbol{u}$ has zero area, or $\boldsymbol{u} \wedge \boldsymbol{u} = \mathbf{0}$
  note it is $\mathbf{0}$ (denoting a bivector) not 0 (denoting scalar).

- Property of bivector: oriented
  - Skew-commutative / anti-commutative : $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

- Basic computation on basis
  - $\boldsymbol{e}_1 \wedge \boldsymbol{e}_1 = \mathbf{0}$
  - $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = -\boldsymbol{e}_2 \wedge \boldsymbol{e}_1$

More on the wedge product

- Properties of wedge product
  - distributive: $(\boldsymbol{u}+\boldsymbol{v})\wedge\boldsymbol{w}=\boldsymbol{u}\wedge\boldsymbol{w}+\boldsymbol{v}\wedge\boldsymbol{w}$ and $\boldsymbol{u}\wedge(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\wedge\boldsymbol{w}+\boldsymbol{u}\wedge\boldsymbol{v}$.
  - associative: $\boldsymbol{u}\wedge\boldsymbol{v}\wedge\boldsymbol{w}=(\boldsymbol{u}\wedge\boldsymbol{v})\wedge\boldsymbol{w}=\boldsymbol{u}\wedge(\boldsymbol{v}\wedge\boldsymbol{w})$.
  - skew-commutative: $\boldsymbol{u}\wedge\boldsymbol{v}=-\boldsymbol{v}\wedge\boldsymbol{u}$.

- Computation example: $\boldsymbol{u}=a\boldsymbol{e}_1+b\boldsymbol{e}_2$, $\boldsymbol{v}=c\boldsymbol{e}_1+d\boldsymbol{e}_2$,

$$
\begin{aligned}
\boldsymbol{u}\wedge\boldsymbol{v} &= (a\boldsymbol{e}_1+b\boldsymbol{e}_2)\wedge(c\boldsymbol{e}_1+d\boldsymbol{e}_2)\\
&= a\boldsymbol{e}_1\wedge c\boldsymbol{e}_1+a\boldsymbol{e}_1\wedge d\boldsymbol{e}_2+b\boldsymbol{e}_2\wedge c\boldsymbol{e}_1+b\boldsymbol{e}_2\wedge d\boldsymbol{e}_2\\
&= ac\,\underbrace{\boldsymbol{e}_1\wedge\boldsymbol{e}_1}_{=\boldsymbol{0}}+ad\boldsymbol{e}_1\wedge\boldsymbol{e}_2+bc\boldsymbol{e}_2\wedge\boldsymbol{e}_1+bd\,\underbrace{\boldsymbol{e}_2\wedge\boldsymbol{e}_2}_{=\boldsymbol{0}}\\
&= ad\boldsymbol{e}_1\wedge\boldsymbol{e}_2+bc\,\underbrace{\boldsymbol{e}_2\wedge\boldsymbol{e}_1}_{=-\boldsymbol{e}_2\wedge\boldsymbol{e}_1}\\
&= (ad-bc)(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)
\end{aligned}
\tag{1}
$$

Recall that $ad-bc=\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

Geometric product $\boldsymbol{uv} := \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}$

- Example $\boldsymbol{uu}$

$$\boldsymbol{u}^2 = \boldsymbol{uu} = \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{u} \wedge \boldsymbol{u} = \|\boldsymbol{u}\|^2 + \mathbf{0} = \|\boldsymbol{u}\|^2$$

  in this case the output is a scalar.

- Example: if $\boldsymbol{u} \perp \boldsymbol{v}$, then

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v} = 0 + \boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{u} \wedge \boldsymbol{v}$$

  in this case the output is a bivector.

- Geometric product is not commutative nor skew-commutative.

- Properties of geometric product
  - If $\boldsymbol{u}$ is parallel with $\boldsymbol{v}$, then $\boldsymbol{uv} = \boldsymbol{vu}$
  - If $\boldsymbol{u} \perp \boldsymbol{v}$, then $\boldsymbol{uv} = -\boldsymbol{vu}$.

More on the geometric product

- Computation example: $\boldsymbol{u} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$, $\boldsymbol{v} = c\boldsymbol{e}_1 + d\boldsymbol{e}_2$,

$$\begin{aligned} \boldsymbol{uv} &= \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v} && \text{def of geometric product} \\ &\overset{(1)}{=} ac + bd + (ad - bc)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \end{aligned}$$

- $ac + bd$ is a scalar, and called grade-0 object.
- $(ad - bc)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$ is a bivector, and called grade-2 object.
- "grade" is like "power", "degree".

Geometric product on $\boldsymbol{e}_1$, $\boldsymbol{e}_2$

- On $\boldsymbol{e}_1$

$$
\begin{aligned}
\boldsymbol{e}_1\boldsymbol{e}_1 &= \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_1 && \text{definition of geometric product} \\
&= \|\boldsymbol{e}_1\|^2 && \boldsymbol{e}_1 \wedge \boldsymbol{e}_1 = \mathbf{0} \\
&= 1 && \text{definition of } \boldsymbol{e}_1
\end{aligned}
$$

  so we have

$$
\boldsymbol{e}_1^2 = 1 \qquad \text{and similarly} \qquad \boldsymbol{e}_2^2 = 1.
$$

- On $\boldsymbol{e}_1\boldsymbol{e}_2$

$$
\begin{aligned}
\boldsymbol{e}_1\boldsymbol{e}_2 &= \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 && \text{definition of geometric product} \\
&= \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 && \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0
\end{aligned}
$$

  Similarly,

$$
\begin{aligned}
\boldsymbol{e}_2\boldsymbol{e}_1 &= \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 + \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 && \text{definition of geometric product} \\
&= \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 && \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0 \\
&= -\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 && \text{wedge product is skew-commutative} \\
&= -\boldsymbol{e}_1\boldsymbol{e}_2
\end{aligned}
$$

Geometric product on $\boldsymbol{e}_1$, $\boldsymbol{e}_2$

- Now we have

$$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1 \qquad \text{and} \qquad \boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_2\boldsymbol{e}_1 \tag{$*$}$$

- Consider $(\boldsymbol{e}_1\boldsymbol{e}_2)^2$

$$\begin{aligned}
(\boldsymbol{e}_1\boldsymbol{e}_2)^2 &= \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 && \text{definition of geometric product} \\
&= -\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2 && \text{by } (*) \\
&= -\boldsymbol{e}_2(\boldsymbol{e}_1\boldsymbol{e}_1)\boldsymbol{e}_2 && \text{geometric product is associative} \\
&= -\boldsymbol{e}_2 1 \boldsymbol{e}_2 && \text{by } (*) \\
&= -\boldsymbol{e}_2\boldsymbol{e}_2 && \\
&= -1 && \text{by } (*)
\end{aligned}$$

Hence

$$(\boldsymbol{e}_1\boldsymbol{e}_2)^2 = -1.$$

- Recall $i^2 = -1$ in complex number: $\boldsymbol{e}_1\boldsymbol{e}_2$ behaves like i, so denote $\boldsymbol{e}_1\boldsymbol{e}_2$ as $\boldsymbol{I}$.

$\mathbb{G}(\mathbb{R}^2)$

- The objects in $\mathbb{G}(\mathbb{R}^2)$
  - (Grade-0) Scalar: 1
  - (Grade-1) Vector: $\boldsymbol{e}_1, \boldsymbol{e}_2$
  - (Grade-2) Bivector: $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{I}$
- There are in total 4 basis, hence $\dim\left(\mathbb{G}(\mathbb{R}^2)\right) = 4 = 2^2$. That is, Clifford algebra of $\mathbb{R}^2$ is an abstract vector space with dimension 4.
- The standard basis of $\mathbb{G}(\mathbb{R}^2)$ is

$$\left\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{I}\right\},$$

any objective in such abstract vector space $\mathbb{G}(\mathbb{R}^2)$, called *multi-vector*, can be expressed as a linear combination of these objects. That is, for all multi-vector $\boldsymbol{v} \in \mathbb{G}(\mathbb{R}^2)$,

$$\boldsymbol{v} = a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{I}.$$

for some scalars $a, b, c, d \in \mathbb{R}$.

The three products

- The three products
  - Dot product $\boldsymbol{u} \cdot \boldsymbol{v}$
  - Wedge product $\boldsymbol{u} \wedge \boldsymbol{v}$
  - Geometric product $\boldsymbol{uv}$
- Relations

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}, \quad \boldsymbol{u} \cdot \boldsymbol{v} = \frac{\boldsymbol{uv} + \boldsymbol{vu}}{2}, \quad \boldsymbol{u} \wedge \boldsymbol{v} = \frac{\boldsymbol{uv} - \boldsymbol{vu}}{2}$$

Geometric-multiplicative inverse

- Given a non-zero multivector $\boldsymbol{u}$

$$\boldsymbol{u}^2 = \boldsymbol{u}\boldsymbol{u} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{u}}_{=\|\boldsymbol{u}\|^2} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{u}}_{=\mathbf{0}} = \|\boldsymbol{u}\|^2.$$

  As $\|\boldsymbol{u}\|^2$ is a non-zero scalar, its scalar inverse exists, and hence

$$\boldsymbol{u}\boldsymbol{u}(\|\boldsymbol{u}\|^2)^{-1} = \|\boldsymbol{u}\|^2(\|\boldsymbol{u}\|^2)^{-1}$$

  which leads to $\dfrac{\boldsymbol{u}\boldsymbol{u}}{\|\boldsymbol{u}\|^2} = 1$. If we veiw $\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$ as a vector $\boldsymbol{v}$, then

$$\boldsymbol{v}\boldsymbol{u} = \boldsymbol{u}\boldsymbol{v} = 1.$$

  That is, $\boldsymbol{v}$ is the geometric-multiplicative inverse of $\boldsymbol{u}$, denoted as $\boldsymbol{u}^{-1}$

$$\boldsymbol{u}^{-1} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}.$$

- Property of geometric-multiplicative inverse: if $\boldsymbol{u}$ has unit norm, i.e., $\|\boldsymbol{u}\| = 1$, then we have $\boldsymbol{u}^{-1} = \boldsymbol{u}$.

Small summary

- Dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ and wedge product $\boldsymbol{u} \wedge \boldsymbol{v}$
- Dot product is commutative, wedge product is skew-commutative
- Geometric product $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}$
- $\mathbb{G}(\mathbb{R}^2)$ basis: $\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 \wedge \boldsymbol{e}_2\}$
- Multivector in $\mathbb{G}(\mathbb{R}^2)$: $\boldsymbol{v} = a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$
- Grade
- Geometric-multiplicative inverse: $\boldsymbol{u}^{-1} = \dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$

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Objects in $\mathbb{G}(\mathbb{R}^2)$, recall

- There are 4 basis and $\dim\left(\mathbb{G}(\mathbb{R}^2)\right) = 2^2$. The standard basis is

$$\left\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{I}\right\},$$

  - (Grade-0) Scalar: 1
  - (Grade-1) Vector: $\boldsymbol{e}_1, \boldsymbol{e}_2$
  - (Grade-2) Bivector: $\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{I}$

- Any multi-vector $\boldsymbol{v} \in \mathbb{G}(\mathbb{R}^2)$

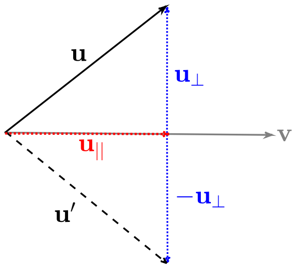
$$\boldsymbol{v} = a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{I}$$

for some scalars $a, b, c, d \in \mathbb{R}$.

Operations in $\mathbb{G}(\mathbb{R}^2)$, recall

- Dot product $\boldsymbol{u} \cdot \boldsymbol{v}$
  Dot product is commutative: $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- Wedge product $\boldsymbol{u} \wedge \boldsymbol{v}$
  Wedge product is skew-commutative: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$
- Geometric product $\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}$
  Geometric product is not commutative nor skew-commutative
- Geometric-multiplicative inverse $\boldsymbol{v}^{-1} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|^2}$
  $\boldsymbol{v}^{-1}$ exists for non-zero $\boldsymbol{v}$
  If $\boldsymbol{v}$ has unit norm, then $\boldsymbol{v}^{-1} = \boldsymbol{v}$
- Dot product produces a scalar, wedge product produces a bivector, geometric product produces a multivector.

Geometric operations on two vectors



- $\boldsymbol{u}_{||}$: the projection of $\boldsymbol{u}$ on $\boldsymbol{v}$

$$\boldsymbol{u}_{||} = (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1}.$$

- $\boldsymbol{u}_{\perp}$: the rejection of $\boldsymbol{u}$ on $\boldsymbol{v}$

$$\boldsymbol{u}_{\perp} = (\boldsymbol{u} \wedge \boldsymbol{v})\boldsymbol{v}^{-1}.$$

- $\boldsymbol{u}'$: the reflection of $\boldsymbol{u}$ on $\boldsymbol{v}$

$$\boldsymbol{u}' = \boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v}.$$

Proofs

- $\boldsymbol{u}_{||} = (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1}$

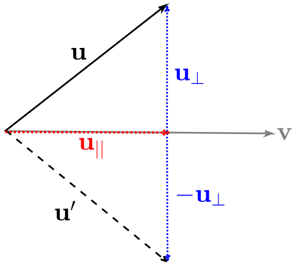
$$
\begin{array}{rcll}
\boldsymbol{uv} & = & \boldsymbol{u}_{||}\boldsymbol{v} + \boldsymbol{u}_{\perp}\boldsymbol{v} & \\
 & = & \boldsymbol{u}_{||}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}_{\perp} & \boldsymbol{u}_{\perp} \perp \boldsymbol{v} \\
 & = & \boldsymbol{u}_{||}\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{u} - \boldsymbol{u}_{||}) & \boldsymbol{u} = \boldsymbol{u}_{||} + \boldsymbol{u}_{\perp} \\
 & = & \boldsymbol{u}_{||}\boldsymbol{v} - \boldsymbol{vu} - \boldsymbol{v}\boldsymbol{u}_{||} & \text{geometric product is distributive} \\
 & = & 2\boldsymbol{u}_{||}\boldsymbol{v} - \boldsymbol{vu} & \boldsymbol{u}_{||} \parallel \boldsymbol{v}
\end{array}
$$

$$
\begin{array}{rcll}
\boldsymbol{u}_{||}\boldsymbol{v} & = & \dfrac{\boldsymbol{uv} + \boldsymbol{vu}}{2} & \text{rearrange} \\
 & = & \boldsymbol{u} \cdot \boldsymbol{v} & \boldsymbol{u} \cdot \boldsymbol{v} = \dfrac{\boldsymbol{uv} + \boldsymbol{vu}}{2} \\
\boldsymbol{u}_{||} & = & (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1} & \boldsymbol{v}^{-1} \text{ exists for non-zero } \boldsymbol{v}
\end{array}
$$

- $\boldsymbol{u}_{\perp} = (\boldsymbol{u} \wedge \boldsymbol{v})\boldsymbol{v}^{-1}$

$$
\begin{array}{rcll}
\boldsymbol{u}_{\perp} = \boldsymbol{u} - \boldsymbol{u}_{||} & = & \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1} & \\
 & = & \boldsymbol{u}\boldsymbol{v}\boldsymbol{v}^{-1} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1} & \boldsymbol{v} \text{ non-zero} \\
 & = & (\boldsymbol{uv} - \boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1} & \text{factorize } \boldsymbol{v}^{-1} \text{ out} \\
 & = & (\boldsymbol{u} \wedge \boldsymbol{v})\boldsymbol{v}^{-1} & \boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}
\end{array}
$$

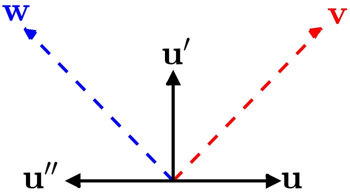
Proofs



- $\boldsymbol{u}' = \boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v}$

$$
\begin{aligned}
\boldsymbol{u}' &= \boldsymbol{u}_{||} - \boldsymbol{u}_\perp \\
\boldsymbol{v}\boldsymbol{u}' &= \boldsymbol{v}\boldsymbol{u}_{||} - \boldsymbol{v}\boldsymbol{u}_\perp \\
&= \boldsymbol{u}_{||}\boldsymbol{v} + \boldsymbol{u}_\perp\boldsymbol{v} && \boldsymbol{u}_{||} \parallel \boldsymbol{v},\ \boldsymbol{u}_\perp \perp \boldsymbol{v} \\
&= (\boldsymbol{u}_{||} + \boldsymbol{u}_\perp)\boldsymbol{v} \\
&= \boldsymbol{u}\boldsymbol{v} && \boldsymbol{u} = \boldsymbol{u}_{||} + \boldsymbol{u}_\perp \\
\boldsymbol{u}' &= \boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v} && \boldsymbol{v}^{-1} \text{ exists for non-zero } \boldsymbol{v}
\end{aligned}
$$

Composite reflections $\boldsymbol{u}'' = (\boldsymbol{v}\boldsymbol{w})^{-1}\boldsymbol{u}(\boldsymbol{v}\boldsymbol{w})$



- $\boldsymbol{u}$ reflect along $\boldsymbol{v}$: $\boldsymbol{u} = \boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v}$.
- $\boldsymbol{u}'$ reflect along $\boldsymbol{w}$: $\boldsymbol{u}'' = \boldsymbol{w}^{-1}\boldsymbol{u}'\boldsymbol{w}$.
- Composite
$$\boldsymbol{u}'' = \boldsymbol{w}^{-1}\boldsymbol{u}'\boldsymbol{w} = \boldsymbol{w}^{-1}\boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v}\boldsymbol{w} = (\boldsymbol{v}\boldsymbol{w})^{-1}\boldsymbol{u}(\boldsymbol{v}\boldsymbol{w}).$$
- On $(\boldsymbol{v}\boldsymbol{w})^{-1} = \boldsymbol{w}^{-1}\boldsymbol{v}^{-1}$
  - $\boldsymbol{w}^{-1}\boldsymbol{v}^{-1}\boldsymbol{v}\boldsymbol{w} = \boldsymbol{w}^{-1}\boldsymbol{w} = 1$, so $\boldsymbol{w}^{-1}\boldsymbol{v}^{-1}$ is the left inverse of $\boldsymbol{v}\boldsymbol{w}$
  - $\boldsymbol{v}\boldsymbol{w}\boldsymbol{w}^{-1}\boldsymbol{v}^{-1} = \boldsymbol{v}^{-1}\boldsymbol{v} = 1$, so $\boldsymbol{w}^{-1}\boldsymbol{v}^{-1}$ is the right inverse of $\boldsymbol{v}\boldsymbol{w}$
  - $\boldsymbol{w}^{-1}\boldsymbol{v}^{-1}$ is both the left and right inverse of $\boldsymbol{v}\boldsymbol{w}$, so $(\boldsymbol{v}\boldsymbol{w})^{-1} = \boldsymbol{w}^{-1}\boldsymbol{v}^{-1}$.

Comparing $\mathbb{G}(\mathbb{R}^2)$ with $\mathbb{C}$: rotate 90-degree

- Object in $\mathbb{G}(\mathbb{R}^2)$: $\boldsymbol{v} = a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{I}$.
  Object in $\mathbb{C}$: $x = a + bi$

- Rotate 90-degree anti-clockwise in $\mathbb{C}$: $y = xi = -b + ai$.
  Rotate 90-degree anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$: $\boldsymbol{u} = \boldsymbol{v}\boldsymbol{I}$

$$\begin{aligned}
\boldsymbol{v}\boldsymbol{I} &= (a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{I})\boldsymbol{I} \\
&= a\boldsymbol{I} + b\boldsymbol{e}_1\boldsymbol{I} + c\boldsymbol{e}_2\boldsymbol{I} + d\boldsymbol{I}^2 \quad \text{geometric product is distributive} \\
&= d - c\boldsymbol{e}_1 + b\boldsymbol{e}_2 + a\boldsymbol{I}
\end{aligned}$$

  where $\boldsymbol{I}^2 = (\boldsymbol{e}_1\boldsymbol{e}_2)^2 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 = -1$, and

$$\boldsymbol{e}_1\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_2, \quad \boldsymbol{e}_2\boldsymbol{I} = \boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 = -\boldsymbol{e}_1.$$

- If $\boldsymbol{v}$ is a vector: $\boldsymbol{v} = b\boldsymbol{e}_1 + c\boldsymbol{e}_2$, we see that $\boldsymbol{u} = \boldsymbol{v}\boldsymbol{I} = -c\boldsymbol{e}_1 + b\boldsymbol{e}_2$, which corresponds to the complex number. Here we see $\boldsymbol{I}(= \boldsymbol{e}_1\boldsymbol{e}_2)$ in $\mathbb{G}(\mathbb{R}^2)$ behaves like i in $\mathbb{C}$.

Comparing $\mathbb{G}(\mathbb{R}^2)$ with $\mathbb{C}$: rotation with any angle

- Rotate θ anti-clockwise in $\mathbb{C}$:

$$y = xe^{i\theta} = (a + bi)(\cos\theta + \sin\theta i) = (a\cos\theta - b\sin\theta) + (a\sin\theta + b\cos\theta)i.$$

- Rotate θ-degree anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$:

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}} \\
&= \boldsymbol{v}(\cos\theta + \sin\theta \boldsymbol{I}) \\
&= (a + b\boldsymbol{e}_1 + c\boldsymbol{e}_2 + d\boldsymbol{I})(\cos\theta + \sin\theta \boldsymbol{I}) \\
&= \begin{array}{cl} & a\cos\theta + a\sin\theta \boldsymbol{I} \\ + & b\boldsymbol{e}_1\cos\theta + b\boldsymbol{e}_1\sin\theta \boldsymbol{I} \\ + & c\boldsymbol{e}_2\cos\theta + c\boldsymbol{e}_2\sin\theta \boldsymbol{I} \\ + & d\boldsymbol{I}\cos\theta + d\boldsymbol{I}\sin\theta \boldsymbol{I} \end{array} = \begin{array}{cl} & a\cos\theta - d\sin\theta \\ + & (b\cos\theta - c\sin\theta)\boldsymbol{e}_1 \\ + & (b\sin\theta + c\cos\theta)\boldsymbol{e}_2 \\ + & (a\sin\theta + d\cos\theta)\boldsymbol{I} \end{array}.
\end{aligned}$$

- If $\boldsymbol{v}$ is a vector: $\boldsymbol{v} = b\boldsymbol{e}_1 + c\boldsymbol{e}_2$, we see $\mathbb{G}(\mathbb{R}^2)$ agrees with $\mathbb{C}$. Here we see $\boldsymbol{I}(= \boldsymbol{e}_1\boldsymbol{e}_2)$ in $\mathbb{G}(\mathbb{R}^2)$ behaves like i in $\mathbb{C}$.

Rotation with any angle, opposite direction

- Rotate θ anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$:
$$\boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}}.$$
- Rotate θ clockwise in $\mathbb{G}(\mathbb{R}^2)$:
$$\boldsymbol{v}\boldsymbol{e}^{-\theta \boldsymbol{I}} \quad \text{or} \quad \boldsymbol{e}^{\theta \boldsymbol{I}}\boldsymbol{v}.$$
- In $\mathbb{C}$, multiplication is commutative: $ae^{i\theta} = e^{i\theta}a$. This is not the case in Clifford algebra, that is
$$\boldsymbol{e}^{\theta \boldsymbol{I}}\boldsymbol{v} \neq \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}},$$
because wedge product is not commutative but skew-commutative
$$\boldsymbol{e}^{\theta \boldsymbol{I}}\boldsymbol{v} = \boldsymbol{e}^{\theta \boldsymbol{I}} \cdot \boldsymbol{v} + \boldsymbol{e}^{\theta \boldsymbol{I}} \wedge \boldsymbol{v} \neq \boldsymbol{v} \cdot \boldsymbol{e}^{\theta \boldsymbol{I}} + \boldsymbol{v} \wedge \boldsymbol{e}^{\theta \boldsymbol{I}} = \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}}.$$

$\boldsymbol{e}^{-\theta \boldsymbol{I}}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}}$ for $\boldsymbol{v}$ is a vector

- Suppose $\boldsymbol{v} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$.

$$
\begin{aligned}
\boldsymbol{e}^{-\theta \boldsymbol{I}}\boldsymbol{v} &= (\cos(-\theta) + \sin(-\theta)\boldsymbol{I})\boldsymbol{v} \\
&= (\cos\theta - \sin\theta \boldsymbol{I})\boldsymbol{v} \\
&= \cos\theta \boldsymbol{v} - \sin\theta \boldsymbol{I}\boldsymbol{v} \\
&= \cos\theta \boldsymbol{v} - \sin\theta \boldsymbol{I}(a\boldsymbol{e}_1 + b\boldsymbol{e}_2) \\
&= \cos\theta \boldsymbol{v} - \sin\theta (a\boldsymbol{I}\boldsymbol{e}_1 + b\boldsymbol{I}\boldsymbol{e}_2) \\
&= \cos\theta \boldsymbol{v} - \sin\theta (-a\boldsymbol{e}_2 + b\boldsymbol{e}_1) \\
&= \cos\theta \boldsymbol{v} + \sin\theta (a\boldsymbol{e}_2 - b\boldsymbol{e}_1) \\
&= \boldsymbol{v}\cos\theta + \sin\theta (a\boldsymbol{e}_1\boldsymbol{I} + b\boldsymbol{e}_2\boldsymbol{I}) \\
&= \boldsymbol{v}\cos\theta + \sin\theta (a\boldsymbol{e}_1 + b\boldsymbol{e}_2)\boldsymbol{I} \\
&= \boldsymbol{v}\cos\theta + \sin\theta \boldsymbol{v}\boldsymbol{I} \\
&= \boldsymbol{v}(\cos\theta + \sin\theta \boldsymbol{I}) \\
&= \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}}.
\end{aligned}
$$

- If $\boldsymbol{v}$ is not a vector but a multivector, $\boldsymbol{e}^{-\theta \boldsymbol{I}}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}}$ in general does not hold. For example $\boldsymbol{v}$ is a scalar v, it is trivial $\boldsymbol{e}^{-\theta \boldsymbol{I}}v \neq v\boldsymbol{e}^{\theta \boldsymbol{I}}$.

Last page - summary

- Projection of vector $\boldsymbol{u}$ on vector $\boldsymbol{v}$:

$$\boldsymbol{u}_{||} = (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}^{-1}.$$

- Rejection of vector $\boldsymbol{u}$ on vector $\boldsymbol{v}$:

$$\boldsymbol{u}_{\perp} = (\boldsymbol{u} \wedge \boldsymbol{v})\boldsymbol{v}^{-1}.$$

- Reflection of vector $\boldsymbol{u}$ on vector $\boldsymbol{v}$:

$$\boldsymbol{u}' = \boldsymbol{v}^{-1}\boldsymbol{u}\boldsymbol{v}$$

- Composite reflections of vector $\boldsymbol{u}$ on vector $\boldsymbol{v}$:

$$\boldsymbol{u}'' = (\boldsymbol{v}\boldsymbol{w})^{-1}\boldsymbol{u}\boldsymbol{v}\boldsymbol{w}$$

- Rotate θ anti-clockwise of vector $\boldsymbol{u}$

$$\boldsymbol{v}\boldsymbol{e}^{\theta \boldsymbol{I}} = \boldsymbol{e}^{-\theta \boldsymbol{I}}\boldsymbol{v}$$

- Note that $\boldsymbol{u}, \boldsymbol{v}$ here are vectors.

End of document