$\mathbb{G}(\mathbb{R}^2)$

Clifford Algebra of $\mathbb{R}^2$

Andersen Ang

Department of Combinatorics and Optimization
Faculty of Mathematics, University of Waterloo
Canada

Homepage: angms.science

First draft: Janurary 29, 2021
Last update: January 31, 2021

Standard basis in $\mathbb{R}^2$

- Vectors $\mathbf{e}_1$ and $\mathbf{e}_2$
  - Both have length 1: $\|\mathbf{e}_1\| = 1$ and $\|\mathbf{e}_2\| = 1$.
  - Orthogonal: $\mathbf{e}_1 \perp \mathbf{e}_2$
- All vector $\mathbf{v} \in \mathbb{R}^2$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{v} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2,$$

where $\alpha, \beta \in \mathbb{R}$ are the *coefficients*, or the *coordinate* of $\mathbf{v}$ in the basis $\mathbf{e}_1, \mathbf{e}_2$.

Dot product $\mathbf{u} \cdot \mathbf{v}$

- A vector-to-scalar operation
  - Input: two vectors $\mathbf{u}, \mathbf{v}$
  - Output: a scalar
- Definition: suppose $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2$. The dot product between $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \cdot \mathbf{v}$, is defined as

$$\mathbf{u} \cdot \mathbf{v} = ac + bd.$$

- Important aspects of dot product
  - Projection: it tells how much a vector is in the same direction to another vector.
  - Orthogonality: if $\mathbf{u} \perp \mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.
- Property of dot product
  - Commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
  - $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

Wedge product $\mathbf{u} \wedge \mathbf{v}$

- A vector-to-bivector operation
  - Input: two vectors $\mathbf{u}, \mathbf{v}$
  - Output: a bivector

- $\mathbf{u} \wedge \mathbf{v}$ is called a bivector.

- What it is: the parallelogram where $\mathbf{u}, \mathbf{v}$ serve as the two sides of it.

- The magnitude of $\mathbf{u} \wedge \mathbf{v}$ is the area of such parallelogram.
  $\implies$ $\mathbf{u} \wedge \mathbf{u}$ has zero area, or $\mathbf{u} \wedge \mathbf{v} = \mathbf{0}$, note that it is $\mathbf{0}$ (denoting a bivector) not 0 (denoting scalar).

- Property of bivector: oriented
  - Skew-commutative / anti-commutative : $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

- Basic computation on basis
  - $\mathbf{e}_1 \wedge \mathbf{e}_1 = \mathbf{0}$
  - $\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1$

More on the wedge product

- Properties of wedge product
  - It is distributive

$$(\mathbf{u}+\mathbf{v})\wedge\mathbf{w}=\mathbf{u}\wedge\mathbf{w}+\mathbf{v}\wedge\mathbf{w}, \quad \text{and} \quad \mathbf{u}\wedge(\mathbf{v}+\mathbf{w})=\mathbf{u}\wedge\mathbf{w}+\mathbf{u}\wedge\mathbf{v}.$$

  - It is associative

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}=(\mathbf{u}\wedge\mathbf{v})\wedge\mathbf{w}=\mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w}).$$

  - It is skew-commutative: $\mathbf{u}\wedge\mathbf{v}=-\mathbf{v}\wedge\mathbf{u}$.
- Computation example: $\mathbf{u}=a\mathbf{e}_1+b\mathbf{e}_2$, $\mathbf{v}=c\mathbf{e}_1+d\mathbf{e}_2$,

$$\begin{aligned}
\mathbf{u}\wedge\mathbf{v} &= (a\mathbf{e}_1+b\mathbf{e}_2)\wedge(c\mathbf{e}_1+d\mathbf{e}_2) \\
&= a\mathbf{e}_1\wedge c\mathbf{e}_1+a\mathbf{e}_1\wedge d\mathbf{e}_2+b\mathbf{e}_2\wedge c\mathbf{e}_1+b\mathbf{e}_2\wedge d\mathbf{e}_2 \\
&= ac\underbrace{\mathbf{e}_1\wedge\mathbf{e}_1}_{=\mathbf{0}}+ad\mathbf{e}_1\wedge\mathbf{e}_2+bc\mathbf{e}_2\wedge\mathbf{e}_1+bd\underbrace{\mathbf{e}_2\wedge\mathbf{e}_2}_{=\mathbf{0}} \\
&= ad\mathbf{e}_1\wedge\mathbf{e}_2+bc\underbrace{\mathbf{e}_2\wedge\mathbf{e}_1}_{=-\mathbf{e}_2\wedge\mathbf{e}_1} \\
&= (ad-bc)(\mathbf{e}_1\wedge\mathbf{e}_2)
\end{aligned} \tag{1}$$

Recall that $ad-bc=\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Geometric product $\mathbf{uv}$

- $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$
- Example

$$\mathbf{u}^2 = \mathbf{uu} = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \wedge \mathbf{u} = \|\mathbf{u}\|^2 + \mathbf{0} = \|\mathbf{u}\|^2$$

  in this case the output is a scalar.

- Example: if $\mathbf{u} \perp \mathbf{v}$, then

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = 0 + \mathbf{u} \wedge \mathbf{v} = \mathbf{u} \wedge \mathbf{v}$$

  in this case the output is a bivector.

- Geometric product is not commutative nor skew-commutative.
- Properties of Geometric product
  - If $\mathbf{u}$ is parallel with $\mathbf{v}$, then $\mathbf{uv} = \mathbf{vu}$
  - If $\mathbf{u} \perp \mathbf{v}$, then $\mathbf{uv} = -\mathbf{vu}$.

More on the geometric product

- Computation example: $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$, $\mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2$,

$$\begin{aligned} \mathbf{uv} &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} && \text{def of geometric product} \\ &= ac + bd + (ad - bc)(\mathbf{e}_1 \wedge \mathbf{e}_2) && \text{from (1)} \end{aligned}$$

- $ac + bd$ is a scalar, and called grade-0 object.
- $(ad - bc)(\mathbf{e}_1 \wedge \mathbf{e}_2)$ is a bivector, and called grade-2 object.
- "grade" is like "power", "degree".

Geometric product on $\mathbf{e}_1$, $\mathbf{e}_2$

- On $\mathbf{e}_1$

$$\begin{aligned}
\mathbf{e}_1\mathbf{e}_1 &= \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_1 && \text{definition of geometric product} \\
&= \|\mathbf{e}_1\|^2 && \mathbf{e}_1 \wedge \mathbf{e}_1 = \mathbf{0} \\
&= 1 && \text{definition of } \mathbf{e}_1
\end{aligned}$$

so we have

$$\mathbf{e}_1^2 = 1 \qquad \text{and similarly} \qquad \mathbf{e}_2^2 = 1.$$

- On $\mathbf{e}_1\mathbf{e}_2$

$$\begin{aligned}
\mathbf{e}_1\mathbf{e}_2 &= \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 && \text{definition of geometric product} \\
&= \mathbf{e}_1 \wedge \mathbf{e}_2 && \mathbf{e}_1 \cdot \mathbf{e}_2 = 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{e}_2\mathbf{e}_1 &= \mathbf{e}_2 \cdot \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_1 && \text{definition of geometric product} \\
&= \mathbf{e}_2 \wedge \mathbf{e}_1 && \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \\
&= -\mathbf{e}_1 \wedge \mathbf{e}_2 && \text{wedge product is skew-commutative} \\
&= -\mathbf{e}_1\mathbf{e}_2
\end{aligned}$$

Geometric product on $\mathbf{e}_1$, $\mathbf{e}_2$

- Now we have

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \qquad \text{and} \qquad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \tag{*}$$

- Consider $(\mathbf{e}_1\mathbf{e}_2)^2$

$$\begin{aligned}
(\mathbf{e}_1\mathbf{e}_2)^2 &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 && \text{definition of geometric product} \\
&= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 && \text{by (*)} \\
&= -\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_1)\mathbf{e}_2 && \text{geometric product is associative} \\
&= -\mathbf{e}_2 1 \mathbf{e}_2 && \text{by (*)} \\
&= -\mathbf{e}_2\mathbf{e}_2 && \\
&= -1 && \text{by (*)}
\end{aligned}$$

Hence

$$(\mathbf{e}_1\mathbf{e}_2)^2 = -1.$$

- Recall $i^2 = -1$ in complex number: $\mathbf{e}_1\mathbf{e}_2$ behaves like i, so denote $\mathbf{e}_1\mathbf{e}_2$ as $\mathbf{I}$.

$\mathbb{G}(\mathbb{R}^2)$

- The objects in $\mathbb{G}(\mathbb{R}^2)$
  - (Grade-0) Scalar: 1
  - (Grade-1) Vector: $\mathbf{e}_1, \mathbf{e}_2$
  - (Grade-2) Bivector: $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{I}$
- There are in total 4 basis, hence $\dim\left(\mathbb{G}(\mathbb{R}^2)\right) = 4 = 2^2$. That is, Clifford algebra of $\mathbb{R}^2$ is an abstract vector space with dimension 4.
- The standard basis of $\mathbb{G}(\mathbb{R}^2)$ is

$$\left\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{I}\right\},$$

any objective in such abstract vector space $\mathbb{G}(\mathbb{R}^2)$, called *multi-vector*, can be expressed as a linear combination of these objects. That is, for all multi-vector $\mathbf{v} \in \mathbb{G}(\mathbb{R}^2)$,

$$\mathbf{v} = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{I}.$$

for some scalars $a, b, c, d \in \mathbb{R}$.

The three products

- The three products
  - Dot product $\mathbf{u} \cdot \mathbf{v}$
  - Wedge product $\mathbf{u} \wedge \mathbf{v}$
  - Geometric product $\mathbf{uv}$

- Relations

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \quad \mathbf{u} \cdot \mathbf{v} = \frac{\mathbf{uv} + \mathbf{vu}}{2}, \quad \mathbf{u} \wedge \mathbf{v} = \frac{\mathbf{uv} - \mathbf{vu}}{2}$$

Geometric-multiplicative inverse

- Given a non-zero multivector $\mathbf{u}$

$$\mathbf{u}^2 = \mathbf{u}\mathbf{u} = \underbrace{\mathbf{u} \cdot \mathbf{u}}_{=\|\mathbf{u}\|^2} + \underbrace{\mathbf{u} \wedge \mathbf{u}}_{=\mathbf{0}} = \|\mathbf{u}\|^2.$$

  As $\|\mathbf{u}\|^2$ is a non-zero scalar, its scalar inverse exists, and hence

$$\mathbf{u}\mathbf{u}(\|\mathbf{u}\|^2)^{-1} = \|\mathbf{u}\|^2(\|\mathbf{u}\|^2)^{-1}$$

  which leads to $\frac{\mathbf{u}\mathbf{u}}{\|\mathbf{u}\|^2} = 1$. If we veiw $\frac{\mathbf{u}}{\|\mathbf{u}\|^2}$ as a vector $\mathbf{v}$, then

$$\mathbf{v}\mathbf{u} = \mathbf{u}\mathbf{v} = 1.$$

  That is, $\mathbf{v}$ is the geometric-multiplicative inverse of $\mathbf{u}$, denoted as $\mathbf{u}^{-1}$

$$\mathbf{u}^{-1} = \frac{\mathbf{u}}{\|\mathbf{u}\|^2}.$$

- Property of geometric-multiplicative inverse: if $\mathbf{u}$ has unit norm, i.e., $\|\mathbf{u}\| = 1$, then we have $\mathbf{u}^{-1} = \mathbf{u}$.

Last page - summary

- Dot product $\mathbf{u} \cdot \mathbf{v}$ and wedge product $\mathbf{u} \wedge \mathbf{v}$
- Dot product is commutative, wedge product is skew-commutative
- Geometric product $\mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$
- $\mathbb{G}(\mathbb{R}^2)$ basis: $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$
- Multivector in $\mathbb{G}(\mathbb{R}^2)$: $\mathbf{v} = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_1 \wedge \mathbf{e}_2$
- Grade
- Geometric-multiplicative inverse: $\mathbf{u}^{-1} = \dfrac{\mathbf{u}}{\|\mathbf{u}\|^2}$

End of document