# Geometric operations in $\mathbb{G}(\mathbb{R}^2)$

## Projection, rejection, reflection and rotation

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# Objects in $\mathbb{G}(\mathbb{R}^2)$

- There are 4 basis and $\dim\left(\mathbb{G}(\mathbb{R}^2)\right) = 2^2$. The standard basis is

$$\left\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{I}\right\},$$

  - (Grade-0) Scalar: $1$
  - (Grade-1) Vector: $\mathbf{e}_1, \mathbf{e}_2$
  - (Grade-2) Bivector: $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{I}$

- Any multi-vector $\mathbf{v} \in \mathbb{G}(\mathbb{R}^2)$

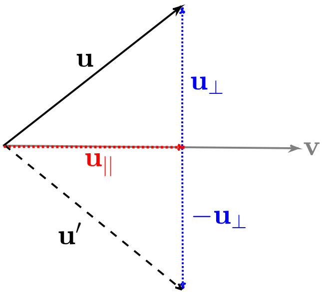
$$\mathbf{v} = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{I}$$

for some scalars $a, b, c, d \in \mathbb{R}$.

# Operations in $\mathbb{G}(\mathbb{R}^2)$

- Dot product $\mathbf{u} \cdot \mathbf{v}$
  Dot product is commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- Wedge product $\mathbf{u} \wedge \mathbf{v}$
  Wedge product is skew-commutative: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$
- Geometric product $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$
  Geometric product is not commutative nor skew-commutative
- Geometric-multiplicative inverse $\mathbf{v}^{-1} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|^2}$
  $\mathbf{v}^{-1}$ exists for non-zero $\mathbf{v}$
  If $\mathbf{v}$ has unit norm, then $\mathbf{v}^{-1} = \mathbf{v}$
- Dot product produces a scalar, wedge product produces a bivector, geometric product produces a multivector.

# Geometric operations on two vectors




- $\mathbf{u}_{||}$: the projection of $\mathbf{u}$ on $\mathbf{v}$

$$\mathbf{u}_{||} = (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1}.$$

- $\mathbf{u}_\perp$: the rejection of $\mathbf{u}$ on $\mathbf{v}$

$$\mathbf{u}_\perp = (\mathbf{u} \wedge \mathbf{v})\mathbf{v}^{-1}.$$

- $\mathbf{u}'$: the reflection of $\mathbf{u}$ on $\mathbf{v}$

$$\mathbf{u}' = \mathbf{v}^{-1}\mathbf{u}\mathbf{v}.$$

# Proofs

- $\mathbf{u}_{||} = (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1}$

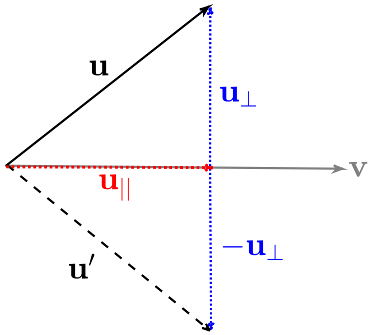
$$\begin{aligned}
\mathbf{uv} &= \mathbf{u}_{||}\mathbf{v} + \mathbf{u}_{\perp}\mathbf{v} \\
&= \mathbf{u}_{||}\mathbf{v} - \mathbf{v}\mathbf{u}_{\perp} && \mathbf{u}_{\perp} \perp \mathbf{v} \\
&= \mathbf{u}_{||}\mathbf{v} - \mathbf{v}(\mathbf{u} - \mathbf{u}_{||}) && \mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_{\perp} \\
&= \mathbf{u}_{||}\mathbf{v} - \mathbf{vu} - \mathbf{v}\mathbf{u}_{||} && \text{geometric product is distributive} \\
&= 2\mathbf{u}_{||}\mathbf{v} - \mathbf{vu} && \mathbf{u}_{||} \parallel \mathbf{v}
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}_{||}\mathbf{v} &= \frac{\mathbf{uv} + \mathbf{vu}}{2} && \text{rearrange} \\
&= \mathbf{u} \cdot \mathbf{v} && \mathbf{u} \cdot \mathbf{v} = \frac{\mathbf{uv} + \mathbf{vu}}{2} \\
\mathbf{u}_{||} &= (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1} && \mathbf{v}^{-1} \text{ exists for non-zero } \mathbf{v}
\end{aligned}$$

- $\mathbf{u}_{\perp} = (\mathbf{u} \wedge \mathbf{v})\mathbf{v}^{-1}$

$$\begin{aligned}
\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{||} &= \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1} \\
&= \mathbf{u}\mathbf{v}\mathbf{v}^{-1} - (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1} && \mathbf{v} \text{ non-zero} \\
&= (\mathbf{uv} - \mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1} && \text{factorize } \mathbf{v}^{-1} \text{ out} \\
&= (\mathbf{u} \wedge \mathbf{v})\mathbf{v}^{-1} && \mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}
\end{aligned}$$

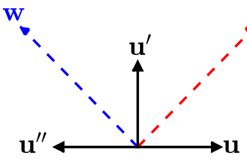
# Proofs




- $\mathbf{u}' = \mathbf{v}^{-1}\mathbf{u}\mathbf{v}$

$$
\begin{aligned}
\mathbf{u}' &= \mathbf{u}_{||} - \mathbf{u}_\perp \\
\mathbf{v}\mathbf{u}' &= \mathbf{v}\mathbf{u}_{||} - \mathbf{v}\mathbf{u}_\perp \\
&= \mathbf{u}_{||}\mathbf{v} + \mathbf{u}_\perp\mathbf{v} && \mathbf{u}_{||} \parallel \mathbf{v},\ \mathbf{u}_\perp \perp \mathbf{v} \\
&= (\mathbf{u}_{||} + \mathbf{u}_\perp)\mathbf{v} \\
&= \mathbf{u}\mathbf{v} && \mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_\perp \\
\mathbf{u}' &= \mathbf{v}^{-1}\mathbf{u}\mathbf{v} && \mathbf{v}^{-1} \text{ exists for non-zero } \mathbf{v}
\end{aligned}
$$

# Composite reflections $\mathbf{u}'' = (\mathbf{v}\mathbf{w})^{-1}\mathbf{u}(\mathbf{v}\mathbf{w})$




- $\mathbf{u}$ reflect along $\mathbf{v}$: $\mathbf{u} = \mathbf{v}^{-1}\mathbf{u}\mathbf{v}$.
- $\mathbf{u}'$ reflect along $\mathbf{w}$: $\mathbf{u}'' = \mathbf{w}^{-1}\mathbf{u}'\mathbf{w}$.
- Composite

$$\mathbf{u}'' = \mathbf{w}^{-1}\mathbf{u}'\mathbf{w} = \mathbf{w}^{-1}\mathbf{v}^{-1}\mathbf{u}\mathbf{v}\mathbf{w} = (\mathbf{v}\mathbf{w})^{-1}\mathbf{u}(\mathbf{v}\mathbf{w}).$$

- On $(\mathbf{v}\mathbf{w})^{-1} = \mathbf{w}^{-1}\mathbf{v}^{-1}$
  - $\mathbf{w}^{-1}\mathbf{v}^{-1}\mathbf{v}\mathbf{w} = \mathbf{w}^{-1}\mathbf{w} = 1$, so $\mathbf{w}^{-1}\mathbf{v}^{-1}$ is the left inverse of $\mathbf{v}\mathbf{w}$
  - $\mathbf{v}\mathbf{w}\mathbf{w}^{-1}\mathbf{v}^{-1} = \mathbf{v}^{-1}\mathbf{v} = 1$, so $\mathbf{w}^{-1}\mathbf{v}^{-1}$ is the right inverse of $\mathbf{v}\mathbf{w}$
  - $\mathbf{w}^{-1}\mathbf{v}^{-1}$ is both the left and right inverse of $\mathbf{v}\mathbf{w}$, so $(\mathbf{v}\mathbf{w})^{-1} = \mathbf{w}^{-1}\mathbf{v}^{-1}$.

# Comparing $\mathbb{G}(\mathbb{R}^2)$ with $\mathbb{C}$: rotate 90-degree

- Object in $\mathbb{G}(\mathbb{R}^2)$: $\mathbf{v} = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{I}$.
  Object in $\mathbb{C}$: $x = a + bi$

- Rotate 90-degree anti-clockwise in $\mathbb{C}$: $y = xi = -b + ai$.
  Rotate 90-degree anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$ : $\mathbf{u} = \mathbf{vI}$

$$\begin{aligned} \mathbf{vI} &= (a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{I})\mathbf{I} \\ &= a\mathbf{I} + b\mathbf{e}_1\mathbf{I} + c\mathbf{e}_2\mathbf{I} + d\mathbf{I}^2 \quad \text{geometric product is distributive} \\ &= d - c\mathbf{e}_1 + b\mathbf{e}_2 + a\mathbf{I} \end{aligned}$$

  where $\mathbf{I}^2 = (\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1$, and

$$\mathbf{e}_1\mathbf{I} = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2, \quad \mathbf{e}_2\mathbf{I} = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1.$$

- If $\mathbf{v}$ is a vector: $\mathbf{v} = b\mathbf{e}_1 + c\mathbf{e}_2$, we see that $\mathbf{u} = \mathbf{vI} = -c\mathbf{e}_1 + b\mathbf{e}_2$, which corresponds to the complex number. Here we see $\mathbf{I}(= \mathbf{e}_1\mathbf{e}_2)$ in $\mathbb{G}(\mathbb{R}^2)$ behaves like $i$ in $\mathbb{C}$.

## Comparing $\mathbb{G}(\mathbb{R}^2)$ with $\mathbb{C}$: rotation with any angle

- Rotate $\theta$ anti-clockwise in $\mathbb{C}$:

$$y = xe^{i\theta} = (a+bi)(\cos\theta+\sin\theta i) = (a\cos\theta-b\sin\theta)+(a\sin\theta+b\cos\theta)i.$$

- Rotate $\theta$-degree anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$:

$$\begin{aligned}\mathbf{u} &= \mathbf{v}\mathbf{e}^{\theta\mathbf{I}} \\ &= \mathbf{v}(\cos\theta + \sin\theta\mathbf{I}) \\ &= (a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{I})(\cos\theta + \sin\theta\mathbf{I})\end{aligned}$$

$$= \begin{array}{cl} & a\cos\theta + a\sin\theta\mathbf{I} \\ + & b\mathbf{e}_1\cos\theta + b\mathbf{e}_1\sin\theta\mathbf{I} \\ + & c\mathbf{e}_2\cos\theta + c\mathbf{e}_2\sin\theta\mathbf{I} \\ + & d\mathbf{I}\cos\theta + d\mathbf{I}\sin\theta\mathbf{I}\end{array} = \begin{array}{cl} & a\cos\theta - d\sin\theta \\ + & (b\cos\theta - c\sin\theta)\mathbf{e}_1 \\ + & (b\sin\theta + c\cos\theta)\mathbf{e}_2 \\ + & (a\sin\theta + d\cos\theta)\mathbf{I}\end{array}.$$

- If $\mathbf{v}$ is a vector: $\mathbf{v} = b\mathbf{e}_1 + c\mathbf{e}_2$, we see $\mathbb{G}(\mathbb{R}^2)$ agrees with $\mathbb{C}$. Here we see $\mathbf{I}(= \mathbf{e}_1\mathbf{e}_2)$ in $\mathbb{G}(\mathbb{R}^2)$ behaves like $i$ in $\mathbb{C}$.

# Rotation with any angle, opposite direction

- Rotate $\theta$ anti-clockwise in $\mathbb{G}(\mathbb{R}^2)$:

$$\mathbf{v}\mathbf{e}^{\theta\mathbf{I}}.$$

- Rotate $\theta$ clockwise in $\mathbb{G}(\mathbb{R}^2)$:

$$\mathbf{v}\mathbf{e}^{-\theta\mathbf{I}} \qquad \text{or} \qquad \mathbf{e}^{\theta\mathbf{I}}\mathbf{v}.$$

- In $\mathbb{C}$, multiplication is commutative: $ae^{i\theta} = e^{i\theta}a$. This is not the case in Clifford algebra, that is

$$\mathbf{e}^{\theta\mathbf{I}}\mathbf{v} \neq \mathbf{v}\mathbf{e}^{\theta\mathbf{I}},$$

because wedge product is not commutative but skew-commutative

$$\mathbf{e}^{\theta\mathbf{I}}\mathbf{v} = \mathbf{e}^{\theta\mathbf{I}} \cdot \mathbf{v} + \mathbf{e}^{\theta\mathbf{I}} \wedge \mathbf{v} \neq \mathbf{v} \cdot \mathbf{e}^{\theta\mathbf{I}} + \mathbf{v} \wedge \mathbf{e}^{\theta\mathbf{I}} = \mathbf{v}\mathbf{e}^{\theta\mathbf{I}}.$$

## $\mathbf{e}^{-\theta\mathbf{I}}\mathbf{v} = \mathbf{v}\mathbf{e}^{\theta\mathbf{I}}$ for $\mathbf{v}$ is a vector

- Suppose $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2$.

$$
\begin{aligned}
\mathbf{e}^{-\theta\mathbf{I}}\mathbf{v} &= (\cos(-\theta) + \sin(-\theta)\mathbf{I})\mathbf{v} \\
&= (\cos\theta - \sin\theta\mathbf{I})\mathbf{v} \\
&= \cos\theta\mathbf{v} - \sin\theta\mathbf{I}\mathbf{v} \\
&= \cos\theta\mathbf{v} - \sin\theta\mathbf{I}(a\mathbf{e}_1 + b\mathbf{e}_2) \\
&= \cos\theta\mathbf{v} - \sin\theta(a\mathbf{I}\mathbf{e}_1 + b\mathbf{I}\mathbf{e}_2) \\
&= \cos\theta\mathbf{v} - \sin\theta(-a\mathbf{e}_2 + b\mathbf{e}_1) \\
&= \cos\theta\mathbf{v} + \sin\theta(a\mathbf{e}_2 - b\mathbf{e}_1) \\
&= \mathbf{v}\cos\theta + \sin\theta(a\mathbf{e}_1\mathbf{I} + b\mathbf{e}_2\mathbf{I}) \\
&= \mathbf{v}\cos\theta + \sin\theta(a\mathbf{e}_1 + b\mathbf{e}_2)\mathbf{I} \\
&= \mathbf{v}\cos\theta + \sin\theta\mathbf{v}\mathbf{I} \\
&= \mathbf{v}(\cos\theta + \sin\theta\mathbf{I}) \\
&= \mathbf{v}\mathbf{e}^{\theta\mathbf{I}}.
\end{aligned}
$$

- If $\mathbf{v}$ is not a vector but a multivector, $\mathbf{e}^{-\theta\mathbf{I}}\mathbf{v} = \mathbf{v}\mathbf{e}^{\theta\mathbf{I}}$ in general does not hold. For example $\mathbf{v}$ is a scalar $v$, it is trivial $\mathbf{e}^{-\theta\mathbf{I}}v \neq v\mathbf{e}^{\theta\mathbf{I}}$.

# Last page - summary

- Projection of vector $\mathbf{u}$ on vector $\mathbf{v}$:

$$\mathbf{u}_{||} = (\mathbf{u} \cdot \mathbf{v})\mathbf{v}^{-1}.$$

- Rejection of vector $\mathbf{u}$ on vector $\mathbf{v}$:

$$\mathbf{u}_{\perp} = (\mathbf{u} \wedge \mathbf{v})\mathbf{v}^{-1}.$$

- Reflection of vector $\mathbf{u}$ on vector $\mathbf{v}$:

$$\mathbf{u}' = \mathbf{v}^{-1}\mathbf{u}\mathbf{v}$$

- Composite reflections of vector $\mathbf{u}$ on vector $\mathbf{v}$:

$$\mathbf{u}'' = (\mathbf{v}\mathbf{w})^{-1}\mathbf{u}\mathbf{v}\mathbf{w}$$

- Rotate $\theta$ anti-clockwise of vector $\mathbf{u}$

$$\mathbf{v}\mathbf{e}^{\theta\mathbf{I}} = \mathbf{e}^{-\theta\mathbf{I}}\mathbf{v}$$

- Note that $\mathbf{u}, \mathbf{v}$ here are vectors.

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