Laplace Transfrom on LODEs

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2012-11-27

References

Joseph H. Distefano III *Feedback and Control System*

Laplace Transform

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \lim_{N\to\infty}\int_0^N f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

Linear ODEs

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$$

- $m \leq n$ (For transfer function be rational)
- $a_n = 1$ (For transfer function be monic)
- $\left[\dfrac{d^j y(t)}{dt^j}\right]_{t=0} \triangleq y_0^j$, $\left[\dfrac{d^j x(t)}{dt^j}\right]_{t=0} \triangleq x_0^j$ (Short hand symbols of initial conditions)

1. $m = 0$

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = x(t)$$

The Laplace Transform is

$$\sum_{k=0}^{n}\left\{a_k\left(s^kY(s) - \sum_{i=0}^{k-1} s^{k-1-i}y_0^k\right)\right\} = X(s)$$

$$\iff \left(\sum_{k=0}^{n} a_k s^k\right)Y(s) - \sum_{k=0}^{n}\sum_{i=0}^{k-1} a_k s^{k-1-i}y_0^k = X(s)$$

$$\iff Y(s) = \frac{X(s) + \sum_{k=0}^{n}\sum_{i=0}^{k-1} a_k s^{k-1-i}y_0^k}{\sum_{i=0}^{k-1} a_k s^k}$$

Take the inverse Laplace Transfrom

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{X(s)}{\sum_{i=0}^{k-1} a_k s^k}\right\} + \mathcal{L}^{-1}\left\{\frac{\sum_{k=0}^{n}\sum_{i=0}^{k-1} a_k s^{k-1-i}y_0^k}{\sum_{i=0}^{k-1} a_k s^k}\right\}$$

2. General $m > 0$

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$$

The Laplace Transform

$$\sum_{k=0}^{n} \left\{ a_k \left(s^k Y(s) - \sum_{i=0}^{k-1} s^{k-1-i} y_0^k \right) \right\} = \sum_{k=0}^{m} \left\{ b_k \left(s^k X(s) - \sum_{i=0}^{k-1} s^{k-1-i} x_0^k \right) \right\}$$

$$\Longleftrightarrow \quad \left(\sum_{k=0}^{n} a_k s^k \right) Y(s) - \sum_{k=0}^{n} \sum_{i=0}^{k-1} a_k s^{k-1-i} y_0^k = \left(\sum_{k=0}^{m} b_k s^k \right) X(s) - \sum_{k=0}^{m} \sum_{i=0}^{k-1} b_k s^{k-1-i} x_0^k$$

$$\Longleftrightarrow \quad Y(s) = \left(\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} \right) X(s) + \frac{\sum_{k=0}^{n} \sum_{i=0}^{k-1} a_k s^{k-1-i} y_0^k}{\sum_{k=0}^{n} a_k s^k} - \frac{\sum_{k=0}^{m} \sum_{i=0}^{k-1} b_k s^{k-1-i} x_0^k}{\sum_{k=0}^{n} a_k s^k}$$

The time domain solution

$$y(t) = \mathcal{L}^{-1} \left\{ \left(\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} \right) X(s) \right\} + \mathcal{L}^{-1} \left(\frac{\sum_{k=0}^{n} \sum_{i=0}^{k-1} a_k s^{k-1-i} y_0^k}{\sum_{k=0}^{n} a_k s^k} \right) - \mathcal{L}^{-1} \left(\frac{\sum_{k=0}^{m} \sum_{i=0}^{k-1} b_k s^{k-1-i} x_0^k}{\sum_{k=0}^{n} a_k s^k} \right)$$

The first 2 term are forced response. The 3 third term is the free response

3. If all inital conditions are zeros

$$y_0^j = x_0^j = 0$$

For $m = 0$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{X(s)}{\sum_{i=0}^{k-1} a_k s^k} \right\}$$

For $m > 0$

$$y(t) = \mathcal{L}^{-1} \left\{ \left(\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} \right) X(s) \right\}$$

4. Impulse Response

With zero initial conditions, when input is Dirac Pulse, the ouput is the Impulse Response

$$x(t) = \delta(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = 1$$

For $m = 0$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{\sum_{i=0}^{k-1} a_k s^k} \right\}$$

For general case, $m > 0$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}\right\}$$

And

$$\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k} = \frac{Y(z)}{X(z)} = G(s) = \text{The Transfer Function}$$

Thus the Impulse Response is equal to the Inverse Laplace Transfrom of the Transfer Function i.e. The Transfer Function in the time domain

$$\text{Impulse Response} = g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\text{T.F.}\}$$

5. Step Response

With zero initial conditions, when input is Heaviside Unit Step, the ouput is the Step Response

$$x(t) = u(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{1}{s}$$

For $m = 0$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1/s}{\sum_{i=0}^{k-1} a_k s^k}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sum_{i=0}^{k-1} a_k s^{k+1}}\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{\text{T.F.}}{s}\right\}$$

For general case, $m > 0$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}\right\} = \mathcal{L}^{-1}\left\{\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^{k+1}}\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$$

6. Ramp Response

With zero initial conditions, when input is Ramp Function, the ouput is the Step Response

$$x(t) = r(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{1}{s^2}$$

For m=0

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1/s^2}{\sum_{i=0}^{k-1} a_k s^k}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sum_{i=0}^{k-1} a_k s^{k+2}}\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{\text{T.F.}}{s^2}\right\}$$

For general case, m>0

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}\right\} = \mathcal{L}^{-1}\left\{\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^{k+2}}\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{\text{T.F.}}{s^2}\right\}$$

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