State Space of Continuous Linear Time Invariant System

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In the state space model of the dynamic system, three variables are important: the *inputs* $u_i(t)$ changes the internal variables called *states* $x_i(t)$ to produce outputs $y_i(t)$. In general, the for a dynamical system having n internal states $x_1(t), ..., x_n(t)$, r inputs $u_1(t), ..., u_r(t)$ and m outputs $y_1(t), ..., y_m(t)$, the system can be described by the following set of differential eqautions:

General State Space

$$\begin{cases} \dfrac{dx_1(t)}{dt} = f_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \\ \dfrac{dx_2(t)}{dt} = f_2\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \\ \vdots \\ \dfrac{dx_n(t)}{dt} = f_n\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \end{cases}$$

$$\begin{cases} y_1(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \\ y_2(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \\ \vdots \\ y_m(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t), t\right) \end{cases}$$

- There are n functions f_i and m functions g_i, both are functions of $n + r + 1$ variables, they describe the relationship between state variables, inputs and output.
- There are n states, so the system is order n
- There are r inputs and m outputs, so the system is multi-input multi-output (MIMO) system.

If the system is time-invariant

For time invariant system, the functions becomes *independent of time*. That is, for a function $h(x, t)$, it becomes $h(x)$.

Time-invariant State Space

$$\begin{cases} \dfrac{dx_1(t)}{dt} = f_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \\ \dfrac{dx_2(t)}{dt} = f_2\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \\ \vdots \\ \dfrac{dx_n(t)}{dt} = f_n\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \end{cases}$$

$$\begin{cases} y_1(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \\ y_2(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \\ \vdots \\ y_m(t) = g_1\left(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t), ..., u_r(t)\right) \end{cases}$$

If the system is linear

For time invariant system, a linear system means the the functions f and g becomes a *linear operator* on the x and u

Linear Time-invariant State Space

$$\begin{cases} \dfrac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + b_{11}u_1(t) + b_{12}u_2(t) + ... + b_{1r}u_r(t) \\ \dfrac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + b_{21}u_1(t) + b_{22}u_2(t) + ... + b_{2r}u_r(t) \\ \vdots \\ \dfrac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + b_{n1}u_1(t) + b_{n2}u_2(t) + ... + b_{nr}u_r(t) \end{cases}$$

$$\begin{cases} y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + ... + c_{1n}x_n(t) + d_{11}u_1(t) + d_{22}u_2(t) + ... + d_{1r}u_r(t) \\ y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + ... + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + ... + d_{2r}u_r(t) \\ \vdots \\ y_m(t) = c_{m1}x_1(t) + c_{m2}x_2(t) + ... + c_{mn}x_n(t) + d_{m1}u_1(t) + d_{m2}u_2(t) + ... + d_{mr}u_r(t) \end{cases}$$

Vector Notation

Let

$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ be $n-$dimensional state vector

$\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$ be $r-$dimensional input vector

$\mathbf{a}_i = [a_{i1}, a_{i2}, ..., a_{in}]$ be $n-$dimensional coefficients vectors
$\mathbf{b}_i = [b_{i1}, b_{i2}, ..., b_{in}]$ be $n-$dimensional coefficients vectors
$\mathbf{c}_i = [c_{i1}, c_{i2}, ..., c_{in}]$ be $m-$dimensional coefficients vectors
$\mathbf{d}_i = [d_{i1}, d_{i2}, ..., d_{in}]$ be $m-$dimensional coefficients vectors

Then

Linear Time-invariant State Space

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \mathbf{a_1x} + \mathbf{b_1u} \\ \dfrac{dx_2(t)}{dt} = \mathbf{a_2x} + \mathbf{b_2u} \\ \vdots \\ \dfrac{dx_n(t)}{dt} = \mathbf{a_nx} + \mathbf{b_nu} \end{cases} \qquad \begin{cases} y_1(t) = \mathbf{c_1x} + \mathbf{d_1u} \\ y_2(t) = \mathbf{c_2x} + \mathbf{d_2u} \\ \vdots \\ y_m(t) = \mathbf{c_mx} + \mathbf{d_mu} \end{cases}$$

Vector-Matrix Notation

Let

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_n\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_n\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}\mathbf{c}_1\\ \mathbf{c}_2\\ \vdots\\ \mathbf{c}_n\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\\ \vdots\\ \mathbf{d}_n\end{bmatrix},\ \dot{\mathbf{x}}=\frac{d}{dt}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix},\ \mathbf{y}=\begin{bmatrix}y_1(t)\\ y_2(t)\\ \vdots\\ y_m(t)\end{bmatrix}$$

Then

Linear Time-invariant
State Space $\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{B}\mathrm{u}$ $\mathbf{y}=\mathbf{Cx}+\mathbf{Du}$

If the system is initially at rest

Perform Laplace Transform on the system with assumption that the system is initially at rest (zero initial value condition)

$$\mathcal{L}\left[\mathbf{x}\right]=X(s)$$
$$\mathcal{L}\left[\dot{\mathbf{x}}\right]=sX(s)$$

The system becomes

$$sX(s)=AX(s)+BU(s)$$
$$Y(s)=CX(s)+DU(s)$$

Re-arrange

$$(sI-A)\,X(s)=BU(s)$$

$$\rightarrow X(s)=(sI-A)^{-1}\,BU(s)$$

$$\rightarrow Y(s)=C\,(sI-A)^{-1}\,BU(s)+DU(s)$$

Thus the system's $transfer\ function$ $G(s)$ is

$$G(s)\hat{=}\frac{Y(s)}{U(s)}=C\,(sI-A)^{-1}\,B+D$$

or

$$G(s)\hat{=}\frac{Y(s)}{U(s)}=C\frac{\mathrm{adj}\,(sI-A)}{\det\,(sI-A)}B+D$$

And the *characteristic equation*

$$\det(sI-A)=0 \qquad \text{(if there is no cancellation between } C\ \mathrm{adj}(sI-A)B \text{ and } \det(sI-A)\text{)}$$

And the *system poles*

$$\text{system poles} \ = \ \text{eigenvalue of } A$$

From control theory, a system is stable if the poles is in the open left half plane. That is, for a pole $p=a+jb$, a has to be negative and thus

$$\text{system is stable} = \ \text{eigenvalue of } A \text{ has negative real parts}$$

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