On matrix expoential and solving state space model

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First created: 2014. Last update: 2017-Feb-1

Consider $F = \begin{cases} \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} \\ \mathbf{x}(\mathbf{0}) = \mathbf{x_0} \end{cases}$. The solution of F is $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

Verification

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

$$\frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}e^{At}\mathbf{x}_0$$

$$\frac{d}{dt}\mathbf{x} = A\underbrace{e^{At}\mathbf{x}_0}_{\mathbf{x}(t)}$$

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}$$

How to compute e^{At} : (1) by matrix exponential formula

For matrix a square real X ,

$$e^X = \sum_{n=1}^{\infty} \frac{X^n}{n!}$$

Thus

$$e^{At} = \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + ...$$

How to compute e^{At} : (2) by inverse Laplace transform

Consider $\begin{cases} \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} \\ \mathbf{x}(\mathbf{0}) = \mathbf{x_0} \end{cases}$, take Laplace Transform to the system

$$\mathcal{L}\left\{\frac{d}{dt}\mathbf{x}\right\} = \mathcal{L}\left\{\mathbf{A}\mathbf{x}\right\}$$

$$\iff s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s)$$

$$\iff \mathbf{X}(s) = (sI - \mathbf{A})^{-1}\mathbf{x}_0$$

Then, obtain $\mathbf{x}(t)$ by $\mathbf{x}(t) = \mathcal{L}^{-1}\{\mathbf{X}\}$, notice that $\mathbf{x}_0$ is a constant thus it can be taken out in the inverse Laplace transform

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{(sI - \mathbf{A})^{-1}\right\}\mathbf{x}_0$$

Since $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, thus

$$\mathbf{x}(t) = \underbrace{\mathcal{L}^{-1}\left\{(sI - \mathbf{A})^{-1}\right\}}_{e^{At}}\mathbf{x}_0$$

And thus

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - \mathbf{A})^{-1}\right\}$$

How to compute e^{At} : (3) by eigendecomposition

A square matrix A can be expressed as product of W (columns are eigenvectors of A) and Λ(diagonal are eigenvalues of A in decreasing order)

$$A = W\Lambda W^{-1}$$

By using this decomposition,

$$e^{At} = \sum_{n=1}^{\infty}\frac{(At)^n}{n!}$$

$$e^{At} = \sum_{n=1}^{\infty}\frac{\left(W\Lambda W^{-1}t\right)^n}{n!}$$

$$e^{At} = W\left(\sum_{n=1}^{\infty}\frac{(\Lambda t)^n}{n!}\right)W^{-1}$$

$$e^{At} = We^{\Lambda t}W^{-1}$$

Solving State Space model.

Consider $\begin{cases} \dfrac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$, replace t by tau and multiplies $e^{-A\tau}$ to the first equation gives:

$$e^{-A\tau}\frac{d}{d\tau}\mathbf{x} = e^{-A\tau}\mathbf{A}\mathbf{x} + e^{-A\tau}\mathbf{B}\mathbf{u}$$

which is

$$e^{-A\tau}\frac{d}{d\tau}\mathbf{x} - e^{-A\tau}\mathbf{A}\mathbf{x} = e^{-A\tau}\mathbf{B}\mathbf{u}$$

By $\dfrac{dfg}{dx} = fg' + gf'$

$$\frac{d}{d\tau}\left(e^{-A\tau}\mathbf{x}\right) = e^{-A\tau}\mathbf{B}\mathbf{u}$$

Thus

$$e^{-A\tau}\mathbf{x}(\tau) = \int_0^t e^{-A\tau}\mathbf{B}\mathbf{u}d\tau + \mathbf{x}_0$$

At time $\tau = t$

$$e^{-At}\mathbf{x}(t) = \int_0^t e^{-A\tau}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{x}_0$$

$$\mathbf{x}(t) = \int_0^t e^{A(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + e^{At}\mathbf{x}_0$$

And thus

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{\mathbf{A}(\mathbf{t}-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathbf{d}\tau + \mathbf{C}\mathbf{e}^{\mathbf{A}\mathbf{t}}\mathbf{x}_{\mathbf{0}} + \mathbf{D}\mathbf{u}$$

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