Similarity transform, Observability and Controllability

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1 Transformation of State Space system

Consider state space model $\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases}$ and consider transformation $\mathbf{x}$ into $\mathbf{z}$ via a nonsingular matrix T as $\mathbf{x} = T\mathbf{z}$:

$$\begin{cases} T\dot{\mathbf{z}} = \mathbf{ATz} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{CTz} + \mathbf{Du} \end{cases}$$

$$\iff \begin{cases} \dot{\mathbf{z}} = \mathbf{T^{-1}ATz} + \mathbf{T^{-1}Bu} \\ \mathbf{y} = \mathbf{CTz} + \mathbf{Du} \end{cases}$$

Which is

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A'z} + \mathbf{B'u} \\ \mathbf{y} = \mathbf{C'z} + \mathbf{D'u} \end{cases}$$

where $\mathbf{A'} = T^{-1}AT$, $\mathbf{B'} = T^{-1}B$, $\mathbf{C'} = CT$ and $\mathbf{D'} = D$
$\mathbf{A'}$ is a matrix, $\mathbf{B'}, \mathbf{C'}$and $\mathbf{D'}$ are all vectors.

2 Eigen-transformation of State Space system

Consider a special case when T is W, the matrix containing the eigenvectors of A :

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{W^{-1}AWz} + \mathbf{W^{-1}Bu} \\ \mathbf{y} = \mathbf{CWz} + \mathbf{Du} \end{cases}$$

Eigendecomposition. Recall that a square matrix A can be expressed as product of W (columns are eigenvectors of A) and Λ (diagonal are eigenvalues of A in decreasing order)

$$A = W\Lambda W^{-1}$$

$$\iff \Lambda = W^{-1}AW$$

Thus

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{\Lambda z} + \mathbf{W^{-1}Bu} \\ \mathbf{y} = \mathbf{CWz} + \mathbf{Du} \end{cases}$$

Or

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{A'z} + \mathbf{B'u} \\ \mathbf{y} = \mathbf{C'z} + \mathbf{D'u} \end{cases}$$

where $\mathbf{A'} = \Lambda$, $\mathbf{B'} = W^{-1}B$, $\mathbf{C'} = CW$ and $\mathbf{D'} = D$

3 Decoupling of state space

Λ is a diagonal matrix, and B' , C' and D' are vectors, consider $D' = 0$ and $\mathbf{u} = u$

$$\begin{cases} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} b_1^{'} \\ b_2^{'} \\ \vdots \\ b_n^{'} \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_1^{'} \ c_2^{'} \ ... \ c_m^{'} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \end{cases}$$

That is

$$\begin{cases} \dot{z}_k = \lambda_k z_k + b_k u \\ y_k = c_k z_k \end{cases}$$

The equations above are different from the initial state space $\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases}$ in the sense that the states z_k are now *decoupled* : only z_k will affect the rate of change of z_k (the $\dot{z}_k$).

The constant λ_k is the *characteristic modes*, such constants determine the properties of the system, thus modes are very important.

Those modes are *uncontrollable* if $b_i = 0$ and *unobservable* if $c_i = 0$.

Those modes can be both controllable and observable, both not, and mix.

Theorem. *Controllability test*

System is completely controllable iff $M_C = \begin{bmatrix} B \ AB \ A^2B \ ... \ A^{n-1}B \end{bmatrix}$ *has full rank. If* M_C *is not full rank, the rank defect of* M_C *tells the number of uncontrollable modes.*

Theorem. *Observability test*

System is completely observable iff $M_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ *has full rank. If* M_O *is not full rank, the rank defect of* M_O *tells the number of unobservable modes.*

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