Optimal Control: Linear Quadratic Regulator

Andersen Ang

First created: 2014. Last update: 2107-Feb-1

Optimal control problems include :

- Minimize the control action : min $\mathbf{u}(t)$
- Regulate output : minimize the output error $\min \mathbf{y}(t) - \mathbf{y}_0$
- Stabilization at equilibrium point : keep $\mathbf{x}(t)$ close to $\mathbf{x}_0$

Consider the system $\begin{cases} \dfrac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$. The optimal control problem is to find the input $\mathbf{u}(t)$ that minimize the cost function J

$$J = \int_0^T \left[\mathbf{u}'(t)R\mathbf{u}(t) + \mathbf{y}'(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}'(T)F\mathbf{x}\,(T)$$

- Since J has quadratic term $u'Ru$ (u' denote tranpose of u), thus such problem is called Linear Quadratic (LQ) problem.
- Matrix R, Q and F describes the cost in the control process. For example R describes the cost of control action. Q and F describe the penalties of $\mathbf{y}(t)$ and $\mathbf{x}(t)$ away from the desired $\mathbf{y}_0$ and $\mathbf{x}_0$.
- $R > 0$ (positive definite) , $Q, F \geq 0$ (semi-positive definite)

There is two type of problems for different T :

- $T < \infty$: The problem is called Finite Time Horizon Control, the goal is to minimize $J = \int_0^T \left[\mathbf{u}^{\mathbf{T}}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right]$ $\mathbf{x}^T(T)F\mathbf{x}\,(T)$
- $T = \infty$: The problem is called Infinite Time Horizon Control, the goal is to minimize $J = \int_0^\infty \left[\mathbf{u}^{\mathbf{T}}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right.$

The Finite Time Horizon Control
The input $\mathbf{u}(t)$ that minimizes $J = \int_0^T \left[\mathbf{u}^{\mathbf{T}}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)F\mathbf{x}\,(T)$is

$$\mathbf{u}(t) = -R^{-1}B^TP(t)\mathbf{x}(t)$$

And the optimal J is thus

$$J* = \mathbf{x}_0^TP(0)\mathbf{x}_0$$

where P is the solution of the matrix Riccati Differential Equation

$$\dot{P}(t) + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + C^TQC = 0$$

with boundary condition

$$P(T) = F$$

Proof. The Finite Time Horizon control problem

Consider $J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)F\mathbf{x}(T)$

By boundary condition, $P(T) = F$

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)P(T)\mathbf{x}(T)$

Then add and subtract $J*$

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)P(T)\mathbf{x}(T) - J* + J*$

Since $J* = x_0^T P(0)x_0$, thus

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)P(T)\mathbf{x}(T) - \mathbf{x}^T(T)P(0)\mathbf{x}(T) + J*$

Group the $\mathbf{x}^T(T)P\mathbf{x}(T)$ terms

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \mathbf{x}^T(T)\Big[P(T) - P(0)\Big]\mathbf{x}(T) + J*$

Express $\mathbf{x}^T(T)\Big[P(T) - P(0)\Big]\mathbf{x}(T)$ as an integral

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt + \int_0^T \frac{d}{dt}\left(\mathbf{x}^T(t)P(t)\mathbf{x}(t)\right) + J*$

Joint the integral together

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t) + \frac{d}{dt}\left(\mathbf{x}^T(t)P(t)\mathbf{x}(t)\right)\right] dt + J*$

Take derivative

$J = \int_0^T \Big[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t) + \dot{\mathbf{x}}^T(t)P(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{P}(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)\dot{\mathbf{x}}(t)\Big] dt + J*$

Apply the state space model $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t) + \Big(A\mathbf{x} + B\mathbf{u}\Big)^T P(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{P}(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)\Big(A\mathbf{x} + B\mathbf{u}\Big)\right] dt + J*$

Apply the state space moel $\mathbf{y} = C\mathbf{x}$ (and thus $\mathbf{y}^T = \mathbf{x}^T C^T$)

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T C^T QC\mathbf{x} + \Big(A\mathbf{x} + B\mathbf{u}\Big)^T P(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{P}(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)\Big(A\mathbf{x} + B\mathbf{u}\Big)\right] dt + J*$

As transpose is linear $(X+Y)^T = X^T + Y^T$, so $\Big(A\mathbf{x} + B\mathbf{u}\Big)^T = \mathbf{x^T A^T} + \mathbf{u^T B^T}$

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T C^T QC\mathbf{x} + \Big(\mathbf{x^T}A^T + \mathbf{u^T}B^T\Big)P(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{P}(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)\Big(A\mathbf{x} + B\mathbf{u}\Big)\right] dt +$
$J*$

Group all $\mathbf{x}^T(.)\mathbf{x}$ together

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T\Big[C^TQC + A^TP(t) + \dot{P}(t) + P(t)A\Big]\mathbf{x} + \mathbf{u^T}B^TP(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)B\mathbf{u}\right] dt + J*$

Apply the Matrix Riccati Equation, $\dot{P}(t) + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + C^TQC = 0$

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T\Big[P(t)BR^{-1}B^TP(t)\Big]\mathbf{x} + \mathbf{u^T}B^TP(t)\mathbf{x}(t) + \mathbf{x}^T(t)P(t)B\mathbf{u}\right] dt + J*$

Rearrange

$J = \int_0^T \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{u^T}B^TP(t)\mathbf{x}(t) + \mathbf{x}^T\Big[P(t)BR^{-1}B^TP(t)\Big]\mathbf{x} + \mathbf{x}^T(t)P(t)B\mathbf{u}\right] dt + J*$

Perform a tricky factorization

$J = \int_0^T \Big[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\Big]R\Big[\mathbf{u}(t) + R^{-1}B^TP(t)\mathbf{x}(t)\Big]dt + J*$

Since the integrand is positive definite quadratic form , so the integrand ≥ 0, to minimize J , let $\mathbf{u}(t) = -R^{-1}B^TP(t)\mathbf{x}(t)$, so

$J = \int_0^T \Big[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\Big]R\Big[\mathbf{u}(t) + R^{-1}B^TP(t)\mathbf{x}(t)\Big]dt = 0$

Thus min $J = J*$ is the minimum achieved. □

The Infinite Time Horizon Control

The input $\mathbf{u}(t)$ that minimize $J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt$ is

$$\mathbf{u}(t) = -R^{-1}B^TP\mathbf{x}(t)$$

And the optimal J is thus

$$J* = \mathbf{x}_0^T P\mathbf{x}_0$$

where P is the solution of the matrix Algebaric Riccati Equation

$$PA + A^T P - PBR^{-1}B^T P + C^T QC = 0$$

notice that P in this case is a constant and so $\dot{P} = 0$

Note that the Riccati Equation is *symmetric.* (Take transpose gives same equation by replacing P^T by P) , thus P can be assumed to be symmetric.

Proof. The Infinite Time Horizon control problem

Consider $J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{y}^T(t)Q\mathbf{y}(t)\right] dt$

Apply the state space moel $\mathbf{y} = C\mathbf{x}$ (and thus $\mathbf{y}^T = \mathbf{x}^T C^T$)

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T C^T QC\mathbf{x}\right] dt$

Apply the Algebaric Riccati Equation, $PA + A^T P - PBR^{-1}B^T P + C^T QC = 0$

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T \left[-PA - A^T P + PBR^{-1}B^T P\right]\mathbf{x}\right] dt$

Expand

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) - \mathbf{x}^T PA\mathbf{x} - \mathbf{x^T} A^T P\mathbf{x} + \mathbf{x^T} PBR^{-1}B^T P\mathbf{x}\right] dt$

Apply state space model $\dot{\mathbf{x}} = A\mathbf{x} + Bu$

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) - \mathbf{x}^T P\left[\dot{\mathbf{x}} - B\mathbf{u}\right] - \left[\dot{\mathbf{x}}^T - \mathbf{u}^T B^T\right]P\mathbf{x} + \mathbf{x^T} PBR^{-1}B^T P\mathbf{x}\right] dt$

Expand

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) - \mathbf{x}^T P\dot{\mathbf{x}} + \mathbf{x}^T PB\mathbf{u} - \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{u}^T B^T P\mathbf{x} + \mathbf{x^T} PBR^{-1}B^T P\mathbf{x}\right] dt$

Rearrange

$J = \int_0^\infty \left[\mathbf{u^T}(t)R\mathbf{u}(t) + \mathbf{x}^T PB\mathbf{u} + \mathbf{u}^T B^T P\mathbf{x} + \mathbf{x^T} PBR^{-1}B^T P\mathbf{x} - \left(\mathbf{x}^T P\dot{\mathbf{x}} + \dot{\mathbf{x}}^T P\mathbf{x}\right)\right] dt$

Perform a tricky factorization

$J = \int_0^T \left[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\right] R \left[\mathbf{u}(t) + R^{-1}B^T P(t)\mathbf{x}(t)\right] dt - \int_0^\infty \left(\mathbf{x}^T P\dot{\mathbf{x}} + \dot{\mathbf{x}}^T P\mathbf{x}\right) dt$

Evaluate the second integral

$J = \int_0^T \left[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\right] R \left[\mathbf{u}(t) + R^{-1}B^T P(t)\mathbf{x}(t)\right] dt - \left(\mathbf{x}^T(t)P\mathbf{x}(t)\right)_0^\infty$

$J = \int_0^T \left[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\right] R \left[\mathbf{u}(t) + R^{-1}B^T P(t)\mathbf{x}(t)\right] dt - \left(\mathbf{x}^T(\infty)P\mathbf{x}(\infty) - \mathbf{x}^T(0)P\mathbf{x}(0)\right)$

$J = \int_0^T \left[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\right] R \left[\mathbf{u}(t) + R^{-1}B^T P(t)\mathbf{x}(t)\right] dt - \mathbf{x}^T(\infty)P\mathbf{x}(\infty) + \mathbf{x}_0^T P\mathbf{x}_0$

Since $P > 0$ so $\mathbf{x}^T(\infty)P\mathbf{x}(\infty) > 0$ and thus $-\mathbf{x}^T(\infty)P\mathbf{x}(\infty) < 0$.

Also since optimal system is stable, so $\mathbf{x}(\infty) \to 0$

To minimize the J , let $\mathbf{u}(t) = -R^{-1}B^T P(t)\mathbf{x}(t)$, so

$J = \int_0^T \left[\mathbf{u^T}(t) + \mathbf{x}^T(t)P(t)BR^{-1}\right] R \left[\mathbf{u}(t) + R^{-1}B^T P(t)\mathbf{x}(t)\right] dt = 0$

Thus min $J = \mathbf{x}_0^T P\mathbf{x}_0$ is the minimum achieved.

−END−

□