Optimal Control: Linear Quadratic Regulator

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Optimal control problems include:
• Minimize the control action: \( \min u(t) \)
• Regulate output: minimize the output error \( \min y(t) - y_0 \)
• Stabilization at equilibrium point: keep \( x(t) \) close to \( x_0 \)

Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx \\
x(0) &= x_0
\end{align*}
\]

The optimal control problem is to find the input \( u(t) \) that minimize the cost function

\[
J = \int_0^T \left[ u'(t)Ru(t) + y'(t)Qy(t) \right] dt + x'(T)Fx(T)
\]

• Since \( J \) has quadratic term \( u'Ru \) (\( u' \) denote tranpose of \( u \)), thus such problem is called Linear Quadratic (LQ) problem.
• Matrix \( R, Q \) and \( F \) describes the cost in the control process. For example \( R \) describes the cost of control action. \( Q \) and \( F \) describe the penalties of \( y(t) \) and \( x(t) \) away from the desired \( y_0 \) and \( x_0 \).
• \( R > 0 \) (positive definite), \( Q, F \geq 0 \) (semi-positive definite)

There is two type of problems for different \( T \):
• \( T < \infty \): The problem is called Finite Time Horizon Control, the goal is to minimize \( J = \int_0^T \left[ u'(t)Ru(t) + y'(t)Qy(t) \right] dt + x'(T)Fx(T) \)
• \( T = \infty \): The problem is called Infinite Time Horizon Control, the goal is to minimize \( J = \int_0^\infty \left[ u'(t)Ru(t) + y'(t)Qy(t) \right] dt + x'(T)Fx(T) \)

The Finite Time Horizon Control

The input \( u(t) \) that minimizes \( J = \int_0^T \left[ u'(t)Ru(t) + y'(t)Qy(t) \right] dt + x'(T)Fx(T) \) is

\[
u(t) = -R^{-1}B^TP(t)x(t)
\]

And the optimal \( J \) is thus

\[
J^* = x_0^T P(0)x_0
\]

where \( P \) is the solution of the matrix Riccati Differential Equation

\[
P'(t) + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + C^TQC = 0
\]

with boundary condition

\[
P(T) = F
\]
Proof. The Finite Time Horizon control problem
Consider \( J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + x^T(T)Fx(T) \)
By boundary condition, \( P(T) = F \)
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + x^T(T)P(T)x(T)
\]
Then add and subtract \( J^* \)
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + x^T(T)P(T)x(T) - J^* + J^*
\]
Since \( J^* = x_0^T P(0)x_0 \), thus
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + x^T(T)P(T)x(T) - x^T(T)P(0)x(T) + J^*
\]
Group the \( x^T(T)P(T)x(T) \) terms
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + x^T(T) \left[ P(T) - P(0) \right] x(T) + J^*
\]
Express \( x^T(T) \left[ P(T) - P(0) \right] x(T) \) as an integral
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt + \int_0^T \frac{d}{dt} \left( x^T(t)P(t)x(t) \right) dt + J^*
\]
Joint the integral together
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) + \frac{d}{dt} \left( x^T(t)P(t)x(t) \right) \right] dt + J^*
\]
Take derivative
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) + \left( Ax + Bu \right)^T \left( P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)x(t) \right) \right] dt + J^*
\]
Apply the state space model \( \dot{x} = Ax + Bu \)
\[
J = \int_0^T \left[ u^T(t)Ru(t) + y^T(t)Qy(t) + \left( Ax + Bu \right)^T \left( P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)x(t) \right) \right] dt + J^*
\]
As transpose is linear \( (X + Y)^T = X^T + Y^T \), so
\[
\left( Ax + Bu \right)^T = x^T A^T + u^T B^T
\]
\[
J = \int_0^T \left[ u^T(t)Ru(t) + x^T C^T Q C x + \left( x^T A^T + u^T B^T \right) \left( P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)x(t) \right) \right] dt + J^*
\]
Group all \( x^T \)(.)\( x \) together
\[
J = \int_0^T \left[ u^T(t)Ru(t) + x^T C^T Q C x + \left( x^T A^T + u^T B^T \right) \left( P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t)x(t) \right) \right] dt + J^*
\]
Apply the Matrix Riccati Equation, \( \dot{P}(t) + P(t)A + A^T P(t) - P(t)BR^{-1}BT P(t) + C^T Q C = 0 \)
\[
J = \int_0^T \left[ u^T(t)Ru(t) + x^T \left( P(t)BR^{-1}BT P(t) \right) x + u^T B^T P(t)x(t) + x^T(t)P(t)Bu \right] dt + J^*
\]
Rearrange
\[
J = \int_0^T \left[ u^T(t)Ru(t) + u^T BT P(t)x(t) + x^T \left( P(t)BR^{-1}BT P(t) \right) x + x^T(t)P(t)Bu \right] dt + J^*
\]
Perform a tricky factorization
\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] \left[ R \left( u(t) + R^{-1}BT P(t)x(t) \right) \right] dt + J^*
\]
Since the integrand is positive definite quadratic form, so the integrand \( \geq 0 \), to minimize \( J \)
, let \( u(t) = -R^{-1}BT P(t)x(t) \), so
\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}BT P(t)x(t) \right] dt = 0
\]
Thus \( \min J = J^* \) is the minimum achieved.

\[ \square \]

The Infinite Time Horizon Control
The input \( u(t) \) that minimize \( J = \int_0^{\infty} \left[ u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt \) is
\[ u(t) = -R^{-1}BT P(t)x(t) \]
And the optimal \( J \) is thus
\[ J^* = \frac{x_0^T P x_0}{2} \]
The Infinite Time Horizon control problem

Thus

To minimize the

Also since optimal system is stable, so

Since

Evaluate the second integral

Perform a tricky factorization

Where \( P \) is the solution of the matrix Algebraic Riccati Equation

\[
PA + A^TP - PBR^{-1}B^TP + C^TCQ = 0
\]

notice that \( P \) in this case is a constant and so \( \dot{P} = 0 \)

Note that the Riccati Equation is symmetric. (Take transpose gives same equation by replacing \( P^T \) by \( P \)), thus \( P \) can be assumed to be symmetric.

Proof. The Infinite Time Horizon control problem

Consider \( J = \int_0^\infty [u^T(t)Ru(t) + y^T(t)Qy(t)] \) dt

Apply the state space model \( y = Cx \) (and thus \( y^T = x^TC^T \))

\[
J = \int_0^\infty [u^T(t)Ru(t) + x^TC^TCx] \) dt
\]

Apply the Algebraic Riccati Equation, \( PA + A^TP - PBR^{-1}B^TP + C^TCQ = 0 \)

\[
J = \int_0^\infty [u^T(t)Ru(t) + x^T[-PA - A^TP + PBR^{-1}B^TP]x] \) dt
\]

Expand

\[
J = \int_0^\infty [u^T(t)Ru(t) - x^TPAx - x^TA^TPx + x^TPBR^{-1}B^TPx] \) dt
\]

Apply state space model \( \dot{x} = Ax + Bu \)

\[
J = \int_0^\infty [u^T(t)Ru(t) - x^TP\dot{x} - x^TPBu - \dot{x}^TPx + x^TPBR^{-1}B^TPx] \) dt
\]

Expand

\[
J = \int_0^\infty [u^T(t)Ru(t) + x^TP\dot{x} - x^TPBu - \dot{x}^TPx + x^TPBR^{-1}B^TPx + u^TPBR^{-1}B^TPx] \) dt
\]

Rearrange

\[
J = \int_0^\infty [u^T(t)Ru(t) + x^TPBu + u^TPBR^{-1}B^TPx + x^TPBR^{-1}B^TPx - (x^TP\dot{x} + \dot{x}^TPx)] \) dt
\]

Perform a tricky factorization

\[
J = \int_0^\infty \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}B^TP(t)x(t) \right] dt - \int_0^\infty (x^TP\dot{x} + \dot{x}^TPx) dt
\]

Evaluate the second integral

\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}B^TP(t)x(t) \right] dt - (x^T(0)P(0)x(0))
\]

\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}B^TP(t)x(t) \right] dt - (x^T(\infty)x(\infty) - x^T(0)x(0))
\]

\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}B^TP(t)x(t) \right] dt - x^T(\infty)x(\infty) + x^T_0P_0x_0
\]

Since \( P > 0 \) so \( x^T(\infty)x(\infty) > 0 \) and thus \( -x^T(\infty)x(\infty) < 0 \).

Also since optimal system is stable, so \( x(\infty) \rightarrow 0 \)

To minimize the \( J \), let \( u(t) = -R^{-1}B^TP(t)x(t) \), so

\[
J = \int_0^T \left[ u^T(t) + x^T(t)P(t)BR^{-1} \right] R \left[ u(t) + R^{-1}B^TP(t)x(t) \right] dt = 0
\]

Thus min \( J = x^T_0P_0x_0 \) is the minimum achieved.

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