Optimal control law for the Infinite Time Horizon Optimal Regulator

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1 The problem

Consider the state space model of input vector u, state vector x and output vector y

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu \\ y = Cx \end{cases}$$

$$\text{Initial condition } x(0) = x_0$$

With given matrices R and Q, the goal is to find the input $u(t)$ that minimize

$$J = \int_0^{\infty} \left[u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt$$

$$\text{such that} \quad \begin{aligned} R &> 0 \\ Q &\geq 0 \end{aligned}$$

That means the problem is an optimization problem

$$u(t) = \arg\min \int_0^{\infty} \left[u^T(t)Ru(t) + y^T(t)Qy(t) \right] dt$$

$$\text{such that} \quad \begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \\ R &> 0 \\ Q &\geq 0 \end{aligned}$$

The solution can be shown that the optimal control law is

$$u(t) = -R^{-1}B^T Px(t)$$

where the matrix P is the solution of the following Algebraic Riccati Equation

$$PA + A^T P - PBR^{-1}B^T P + C^T QC = 0$$

note that P is positive definite and unique.

The minimal value of J will be

$$J_{min} = x_0^T P x_0$$

2 The proof of optimality and derivation of the optimal law

Consider

$$J = \int_0^\infty \left[u^T(t)Ru(t) + y^T(t)Qy(t)\right] dt$$

Using $y(t) = Cx(t)$ and hence $y^T(t) = x^T(t)C^T$, the J becomes

$$J = \int_0^\infty \left[u^T(t)Ru(t) - x^T(t)C^TQCx(t)\right] dt$$

Using $PA + A^TP - PBR^{-1}B^TP + C^TQC = 0$

$$C^TQC = -\left(PA + A^TP - PBR^{-1}B^TP\right)$$

Thus J becomes

$$J = \int_0^\infty \left[u^T(t)Ru(t) - x^T(t)\left(PA + A^TP - PBR^{-1}B^TP\right)x(t)\right] dt$$

Expand

$$J = \int_0^\infty \left[u^T(t)Ru(t) - x^T(t)PAx(t) - x^T(t)A^TPx(t) + x^T(t)PBR^{-1}B^TPx(t)\right] dt$$

Since $\dot{x} = Ax + Bu$, so $Ax = \dot{x} - Bu$

$$J = \int_0^\infty \left[u^T(t)Ru(t) - x^T(t)P\left(\dot{x}(t) - Bu(t)\right) - \left(\dot{x}^T(t) - u^T(t)B^T\right)Px(t) + x^T(t)PBR^{-1}B^TPx(t)\right] dt$$

$$J = \int_0^\infty \left[u^T(t)Ru(t) - x^T(t)P\dot{x}(t) + x^T(t)PBu(t) - \dot{x}^T(t)Px(t) + u^T(t)B^TPx(t) + x^T(t)PBR^{-1}B^TPx(t)\right] dt$$

$$J = \int_0^\infty \left[u^T(t)Ru(t) + x^T(t)PBu(t) + u^T(t)B^TPx(t) + x^T(t)PBR^{-1}B^TPx(t)\right] dt - \int_0^\infty \left[x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t)\right] dt$$

Use short hand

$$J = \int_0^\infty \left[u^TRu + x^TPBu + u^TB^TPx + x^TPBR^{-1}B^TPx\right] dt - \int_0^\infty \left[x^TP\dot{x} + \dot{x}^TPx\right] dt$$

Now consider the second term

$$\int_0^\infty \left[x^TP\dot{x} + \dot{x}^TPx\right] dt = \left[x^TP\left(\int_0^\infty \dot{x}dt\right) + \left(\int_0^\infty \dot{x}^Tdt\right)Px\right]$$

$$\int_0^\infty \left[x^TP\dot{x} + \dot{x}^TPx\right] dt = 2\left[x^TPx\right]_0^\infty$$

$$\int_0^\infty \left[x^TP\dot{x} + \dot{x}^TPx\right] dt = \left[x(\infty)Px(\infty) - x_0Px_0\right]$$

$x(\infty) = 0$ since the optimal solution is stable

$$\int_0^\infty \left[x^TP\dot{x} + \dot{x}^TPx\right] dt = -x_0Px_0$$

Therefore J becomes

$$J = \int_0^\infty \left[u^T Ru + x^T PBu + u^T B^T Px + x^T PBR^{-1}B^T Px\right] dt + x_0 Px_0$$

Let $x^T PBu = x^T PBR^{-1}Ru$ and $u^T B^T Px =$

$$J = \int^\infty \left[u^T Ru + x^T PBR^{-1}Ru + u^T B^T Px + x^T PBR^{-1}B^T Px\right] dt + x_0 Px_0$$

Now consider $u^T Ru + x^T PBR^{-1}Ru + u^T B^T Px + x^T PBR^{-1}B^T Px$, it is equal to

$$\left(u^T + x^T PBR^{-1}\right) Ru + u^T B^T Px + x^T PBR^{-1}B^T Px$$

$$= \left(u^T + x^T PBR^{-1}\right) Ru + \left(u^T + x^T PBR^{-1}\right) B^T Px$$

$$= \left(u^T + x^T PBR^{-1}\right) \left(Ru + B^T Px\right)$$

$$= \left(u^T + x^T PBR^{-1}\right) R \left(u + R^{-1}B^T Px\right)$$

Therefore

$$J = \int_0^\infty \left[\left(u^T + x^T PBR^{-1}\right) R \left(u^T + R^{-1}B^T Px\right)\right] dt + x_0 Px_0$$

Note that $\left(u + R^{-1}B^T Px\right)^T = \left(u^T + x^T PBR^{-1}\right)$, thus

$$J = \int_0^\infty \underbrace{\left(u^T + x^T PBR^{-1}\right) R \left(u^T + x^T PBR^{-1}\right)^T}_{\text{positive semi-definite}} dt + x_0 Px_0$$

Since $x_0 Px_0$ is independent of u , so it is the lower bound of the J

$$\left(u^T + x^T PBR^{-1}\right) R \left(u^T + x^T PBR^{-1}\right)^T = 0 \iff u = R^{-1}BPx$$

Thus the optimal control law and the optimal value of J are

$$u = R^{-1}BPx \; J_{min} = x_0 Px_0$$

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