Laplace Transfrom on System Modeling

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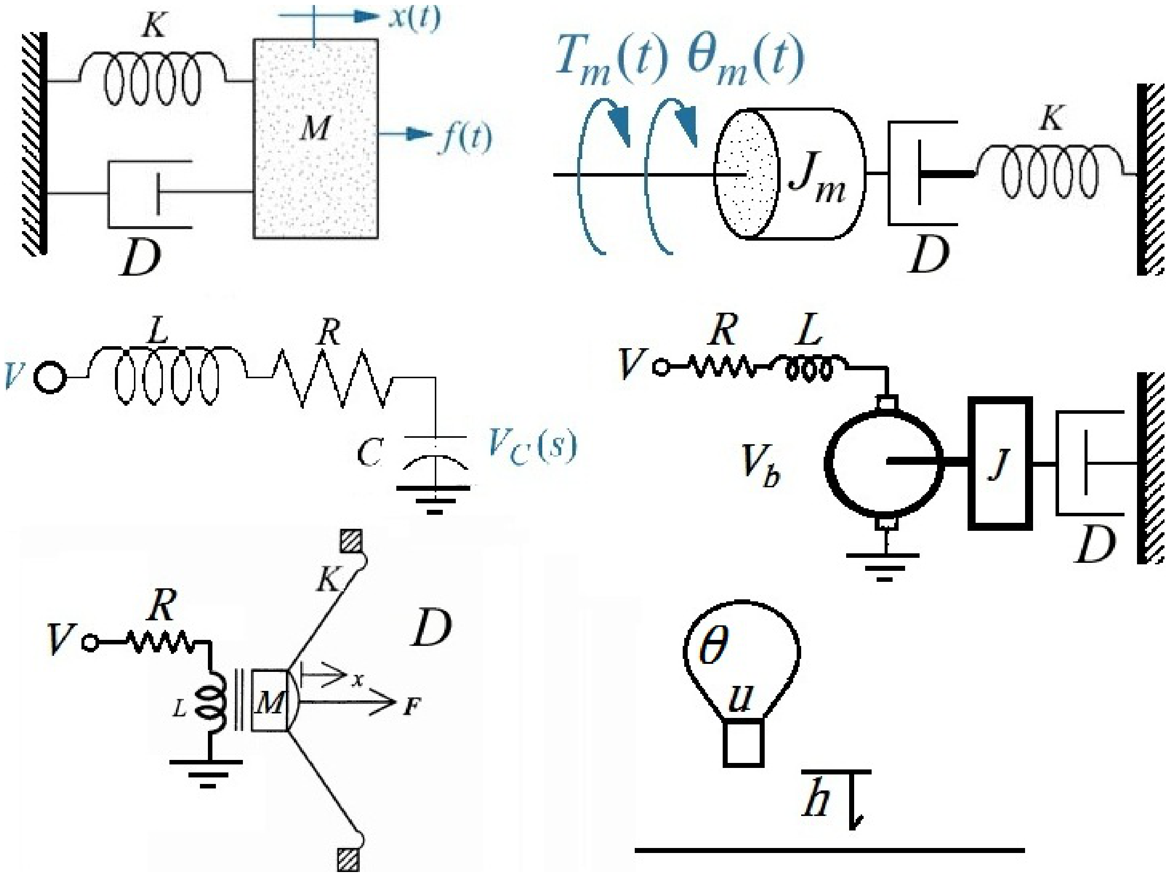

1 Summary

The general steps

- List out the time-domain differential equations from Physical Laws
- Take Laplace Transfrom with zero initial condition (usually)
- Algebraic rearrangement to get the system transfer function
- Take Inverse Laplace Transform to get the time domain response.

2 Mechanical System : Translational Spring-Mass-Damper

Take input $f(t)$, output $x(t)$

Physics of devices :

$$F_{spring} = kx \qquad F_{friction} = Dv \qquad F_{mass} = ma$$

By Newton's Law of Motion and D'Alembert's principle

$$f(t) = M\ddot{x}(t) + D\dot{x}(t) + kx(t)$$

Take Laplace Transfrom

$$F(s) = s^2MX(s) + DsX(s) + kX(s)$$

The T.F. is thus

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Ds + k}$$

3 Mechanical System : Rotational Spring-Mass-Damper

Take the input $T(t)$, output $\theta(t)$

Physics of the devices :

$$T_{mass} = J\alpha \qquad T_{friction} = D\omega \qquad T_{spring} = k\theta$$

By Newton's Law of Motion and D'Alembert's principle

$$T(t) = J\ddot{\theta}(t) + D\dot{\theta}(t) + k\theta(t)$$

Take Laplace Transform

$$T(s) = s^2J\theta(s) + Ds\theta(s) + k\theta(s)$$

The T.F. is thus

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Ds + k}$$

4 Electrical System : LRC Network

Take input $v(t)$, output $v_C(t)$

Physics of the devices :

$$v_R = RI \qquad v_C = \frac{Q}{C} \qquad v_L = L\frac{di}{dt}$$

By Kirchoff's Law

$$v(t) = L\dot{i}(t) + Ri(t) + \frac{1}{C}\int i(t)dt$$

Take Laplace Transfrom

$$V(s) = LsI(s) + RI(s) + \frac{1}{Cs}I(t)$$

Rearrange

$$I(s) = \frac{sC}{LCs^2 + RCs + 1}V(s)$$

$$V_c(s) = \frac{1}{Cs}I(t) = \frac{1}{LCs^2 + RCs + 1}V(s)$$

Thus the T.F. is

$$\frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$

5 Electro-machnical System : Motor

Take input as $v(t)$, output $\theta(t)$

Physics of the devices :

$$T_{Motor} = \alpha I \quad v_{back} = \beta\omega \quad v_L = L\frac{di}{dt} \quad v_R = RI \quad T_{mass} = J\alpha \quad T_{friction} = D\omega$$

By Kirchoff's Law

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + v_{back}(t)$$

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \beta\frac{d\theta(t)}{dt}$$

Take Laplace Transfrom

$$V(s) = RI(s) + sLI(s) + s\beta\theta(s)$$

Rearrange

$$I(t) = \frac{V(s) - s\beta\theta(s)}{R + Ls}$$

By Newton's Law of Motion and D'Alembert's principle

$$T(t) = J\ddot{\theta}(t) + D\dot{\theta}(t)$$

$$T(t) = \alpha i(t)$$

So

$$\alpha i(t) = J\ddot{\theta}(t) + D\dot{\theta}(t)$$

Take Laplace Transfrom

$$\alpha I(s) = s^2 J\theta(s) + Ds\theta(s)$$

Plug in the $I(s)$

$$\alpha \frac{V(s) - s\beta\theta(s)}{R + Ls} = s^2 J\theta(s) + Ds\theta(s)$$

Rearrange

$$\alpha \frac{V(s)}{R + Ls} = \left(s^2 J + Ds + \frac{s\alpha\beta}{R + Ls} \right) \theta(s)$$

The T.F. is thus

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{\alpha}{\left(s^2 J + Ds + \frac{s\alpha\beta}{R + Ls} \right) (R + Ls)} = \frac{\alpha}{LJs^3 + s^2 (RJ + DL) + s (DR + \alpha\beta)}$$

6 Electro-mechanical system : Loudspeaker

Take input as $v(t)$, output as $x(t)$

Physcis of the devices

$$F_B = ilB \qquad F = m\ddot{x} + D\dot{x} + kx \qquad v = Ri + L\frac{di}{dt}$$

Take Laplace Transfroms

$$F_B(s) = I(s)lB \qquad F(s) = s^2 mX(s) + DsX(s) + kX(s) \qquad V(s) = RI(s) + LsI(s)$$

For the force equation and the current equation

$$I(s)lB = s^2 mX(s) + DsX(s) + kX(s) \qquad I(s) = \frac{V(s)}{R + Ls}$$

Combine the eqaution

$$V(s)\frac{lB}{R + Ls} = \left(s^2 m + Ds + k \right) X(s)$$

Rearrange

$$\frac{X(s)}{V(s)} = \frac{1}{ms^2 + Ds + k} \cdot \frac{lB}{R + Ls}$$

Thus, the T.F. is

$$G(s) = \frac{lB}{mLs^3 + (mR + DL) s^2 + (bR + kL) s + Rk}$$

7 Thermo-mechanical System : Hot air Balloon

Take input as u , the heating rate, take output as$h$, the altitude

Physics of the devices :

$$\theta = \text{balloon temperature} \quad \dot{\theta} = -\alpha\theta + u$$

$$F_{drag} = \beta v \qquad F_{up} = r\theta \quad \text{(Buotancy)}$$

Ignore $w = mg$ force of gravity

The euqtions are

$$\begin{cases} \dot{\theta} = -\alpha\theta + u \\ \ddot{h} + \beta h = r\theta \end{cases}$$

Take Laplace Transfrom

$$s\theta(s) = -\alpha\theta(s) + U(s) \qquad\qquad s^2H(s) + \beta H(s) = r\theta(s)$$

Thus

$$\frac{H(s)}{U(s)} = \frac{r}{s^3 + (\alpha+\beta)s^2 + \alpha\beta s}$$

$$-END-$$