Kinematic Analysis on a 2D 2-linked arm

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Robotics problems

- Forward Kinematics Problem : Given the parameters of the joints and the links, find the positions and orientations of the end-effectors.
- Inverse Kinematics Problem : Given the positions and orientations of the end-effectors, determine if it is possible for the robot to achieve such posture, if possible, find all the possible postures.
- Statics Problem : Given the parameters of the joints and the links, find the force to keep the end-effector stay still in a position.
- Forward Dynamic Problem : Given the parameters of the joints and links, find the force / energy on the end-effector.

A 2D case, 2-linked arm will be used to illustrate these ideas.

The Forward Kinematics Problem

Given the link length L_1 , L_2 , joint angles θ_1 , θ_2, find the position of point P (x_2^0, y_2^0)
* x_2^0 means x coordinate of point 2 w.r.t. base frame $o_0(x_0 y_0)$

By simple geometry , the x coordinate of P in base frame is $x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$
And $y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$

$$\begin{cases} x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{cases}$$

Using in matrix-vector notation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \cos(\theta_1+\theta_2) \\ \sin\theta_1 & \sin(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

Thus scuh equation can be used to solve for the Forward problem, given L_1 , L_2 , θ_1 , θ_2, the position x, $y$can be computed.

The Inverse Kinematics Problem

Given L_1 , L_2 , point $P(x, y)$, find the θ_1, θ_2

To find θ_2 , consider the triangle formed by point P, O , and joint 2, apply the cosine-law :

$$\overline{OP}^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\pi - \theta_2)$$

i.e.

$$x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2$$

To find θ_2 , we can thus use

$$\theta_2 = \cos^{-1}\left\{\frac{x^2 + y^2 - (L_1^2 + L_2^2)}{2L_1L_2}\right\}$$

Numerically, because of the poor performance of $\cos^{-1}$, $\sin^{-1}$, $\tan^{-1}$, it is better to use $atan2(y, x)$.

To use $atan2$, we need to find $\sin\theta_2$.

Let $r = \sqrt{x^2 + y^2}$

$$\sin\theta_2 = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left[\frac{r^2 - (L_1^2 + L_2^2)}{2L_1L_2}\right]^2} = \sqrt{\frac{4L_1^2L_2^2 - r^4 + 2r^2(L_1^2 + L_2^2) - (L_1^2 + L_2^2)^2}{4L_1^2L_2^2}}$$

$$\sin\theta_2 = \frac{\sqrt{r^4 + 2r^2(L_1^2 + L_2^2) - (L_1^2 - L_2^2)^2}}{2L_1L_2}$$

Thus

$$\tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2} = \frac{\sqrt{(x^2 + y^2)^2 + 2(x^2 + y^2)^2(L_1^2 + L_2^2) - (L_1^2 - L_2^2)^2}}{x^2 + y^2 - (L_1^2 + L_2^2)}$$

And thus the atan2(y,x) is

$$\texttt{atan2(y,x)} = \begin{cases} \tan^{-1}\left\{\frac{\sqrt{\left(x^2+y^2\right)^2 + 2\left(x^2+y^2\right)^2\left(L_1^2+L_2^2\right) - \left(L_1^2-L_2^2\right)^2}}{x^2+y^2-\left(L_1^2+L_2^2\right)}\right\} & x > 0 \\ \tan^{-1}\left\{\frac{\sqrt{\left(x^2+y^2\right)^2 + 2\left(x^2+y^2\right)^2\left(L_1^2+L_2^2\right) - \left(L_1^2-L_2^2\right)^2}}{x^2+y^2-\left(L_1^2+L_2^2\right)}\right\} + \pi & x < 0\,, y \geq 0 \\ \tan^{-1}\left\{\frac{\sqrt{\left(x^2+y^2\right)^2 + 2\left(x^2+y^2\right)^2\left(L_1^2+L_2^2\right) - \left(L_1^2-L_2^2\right)^2}}{x^2+y^2-\left(L_1^2+L_2^2\right)}\right\} - \pi & x < 0\,, y < 0 \\ +\frac{\pi}{2} & x = 0\,, y > 0 \\ -\frac{\pi}{2} & x = 0\,, y < 0 \\ \text{Undefined} & x = y = 0 \end{cases}$$

After the θ_2 is found, we can find θ_1

$$\theta_1 = \text{atan2}\frac{y}{x} - \theta_2$$

Therefore, the solution is thus

$$\begin{cases} \theta_2 = \text{atan2}\frac{\sqrt{\left(x^2+y^2\right)^2 + 2\left(x^2+y^2\right)^2\left(L_1^2+L_2^2\right) - \left(L_1^2-L_2^2\right)^2}}{x^2+y^2-\left(L_1^2+L_2^2\right)} \\ \theta_1 = \text{atan2}\frac{y}{x} - \theta_2 \end{cases}$$

$$-END-$$