Transformation Matrix I
Rotation Matrix

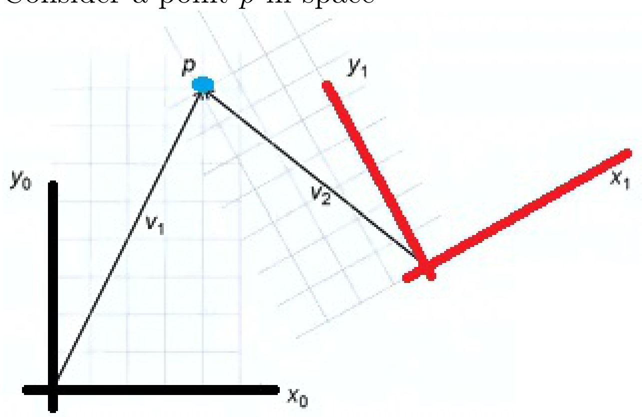
February 22, 2013

In a $n-$link robot arm, what is the position of end effector w.r.t. base reference frame ?

- By geometry inspection, but this method will become very complicated when the system is complex
- By transformation matrix, a systematic method

Review of some vector algebra

Consider a point p in space



The coordinate of p w.r.t. reference frame zero $(x_0o_0y_0)$, denoted as p^0 , is

$$p^0 = 4\hat{x} + 8\hat{y} = (4, 8) = [4, 8]^T$$

And the coordinate of p w.r.t. reference frame 1 $(x_1o_1y_1)$, denoted as p^1 , is

$$p^1 = -3\hat{x} + 7\hat{y} = (-3, 7) = [-3, 7]^T$$

Using vector notation,

$$p^0 = [4, 8]^T = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad p^1 = [-3, 7]^T = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

Similarly we can find the coordinate of origin in frame A w.r.t to frame B

$$\text{Origin}_0 \text{ w.r.t. frame}_1 = o_0^1 = \begin{bmatrix} -10.3 \\ 2 \end{bmatrix}$$

$$\text{Origin}_1 \text{ w.r.t. frame}_0 = o_1^0 = \begin{bmatrix} 10 \\ 3.3 \end{bmatrix}$$

Transforming between Frames

$$p^0 = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad p^1 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

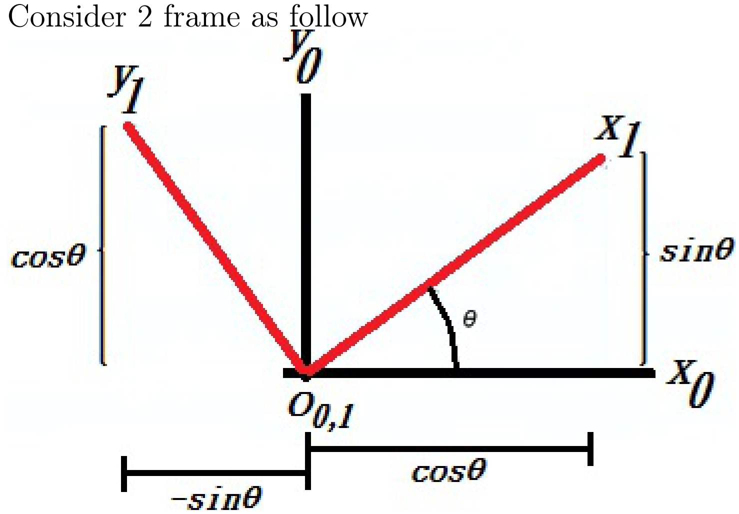
How is p^0 related to p^1 ?

$$p^0 + o_0^1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} -10.3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6.3 \\ 10 \end{bmatrix} \neq p^1 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

Because the frame 1 are rotated, so the 2 vectors are not equal by just addition

Rotation

Consider 2 frame as follow



To express the unit vector $(\hat{x}_1, \hat{y}_1)$ w.r.t frame 0

That is, express x_1 and y_1 using x_0 and y_0 :

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Combine them into one matrix, the *rotation matrix* is thus

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thus, a point p w.r.t frame 1 p^1 and point p w.r.t. frame 0 p^0 are related by

$$p^0 = R_1^0 p^1$$

That is

$$\begin{matrix}\text{p expressed in} \\ \text{frame 0}\end{matrix} = \begin{matrix}\text{unit vector of frame 1} \\ \text{expressed in frame 0}\end{matrix} \times \begin{matrix}\text{p expressed in} \\ \text{frame 1}\end{matrix}$$

$$p^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} p^1$$

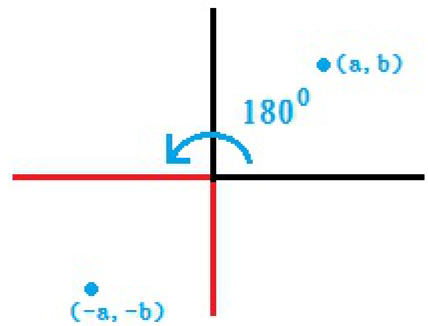
Difference case for special θ

Since θ is the angle between 2 x axis,

When $\theta = 90^o$

$$R_1^0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

That is , first swap(x, y) , then make the new x as negative.



When $\theta = 180^o$

$$R_1^0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

When $\theta = 270^o$ and 360^o , or any other angle, the logic is the same.

Re-visiting the rotation matrix

Since the rotation matrix is obtained by using the x_1^0 and y_1^0 as it's column vector

$$R_1^0 = \left[\begin{pmatrix} x_1^0 \end{pmatrix} \begin{pmatrix} y_1^0 \end{pmatrix} \right]$$

And x_1^0 , y_1^0 is expressed using x_0 , y_0, so they have 2 component

$$x_1^0 = \begin{bmatrix} \text{x-component of } x_1 \text{in frame 0} \\ \text{y component of } x_1 \text{ in frame 0} \end{bmatrix} \qquad y_1^0 = \begin{bmatrix} \text{x-component of } y_1 \text{in frame 0} \\ \text{y component of } y_1 \text{in frame 0} \end{bmatrix}$$

Thus

$$R_1^0 = \left[\begin{pmatrix} \text{x-component of } x_1 \text{in frame 0} \\ \text{y component of } x_1 \text{in frame 0} \end{pmatrix} \begin{pmatrix} \text{x-component of } y_1 \text{in frame 0} \\ \text{y component of } y_1 \text{in frame 0} \end{pmatrix} \right]$$

Recall that, we can extract a vector's component by using **dot product**
That is, for a vector $\bar{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

$$V_x = \bar{V} \cdot \hat{x}$$

Thus

$$R_1^0 = \left[\begin{pmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{pmatrix} \begin{pmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{pmatrix} \right]$$

And therefore, we can use same logic to find R_0^1

$$R_0^1 = \left[\begin{pmatrix} x_0 \cdot x_1 \\ x_0 \cdot y_1 \end{pmatrix} \begin{pmatrix} y_0 \cdot x_1 \\ y_0 \cdot y_1 \end{pmatrix} \right]$$

These method are actually same as the graphical method, in graphical method, the diagram just show the **projection** of the vector component directly in the diagram. But when the diagram is very complicated, it is better to use the dot product method.

Inverse, Transpose

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \qquad R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

Since dot product is commutative, therefore, consider the transpose

$$\left(R_1^0\right)^T = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}^T = \begin{bmatrix} x_1 \cdot x_0 & x_1 \cdot y_0 \\ y_1 \cdot x_0 & y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix} = R_0^1$$

Therefore

$$\left(R_1^0\right)^T = R_0^1$$

But since R_1^0 R_0^1 do the opposite thing, they are inverse to each other

$$\left(R_1^0\right)^{-1} = R_0^1$$

Thus

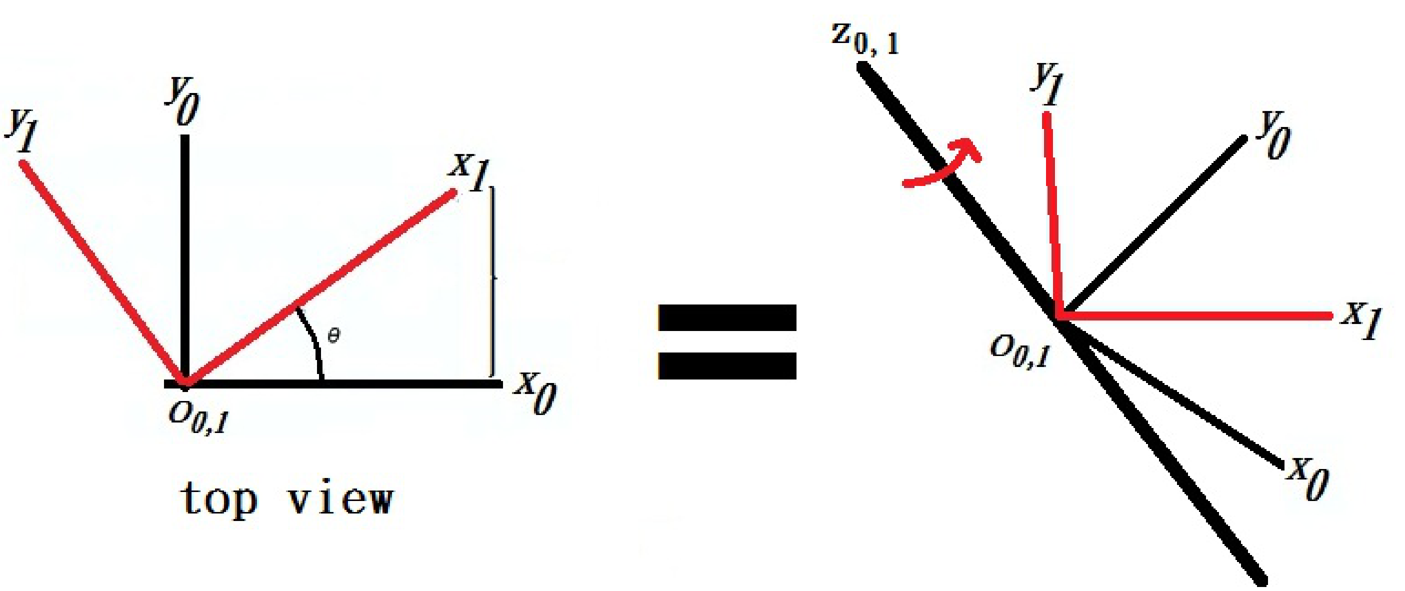
$$\left(R_1^0\right)^{-1} = \left(R_1^0\right)^T$$

Small summary

- $\left(R_i^j\right)^T = R_j^i$
- $\left(R_i^j\right)^{-1} = R_j^i$
- Rotation matrix is **orthogonal :** $\left(R_i^j\right)^{-1} = \left(R_i^j\right)^T$

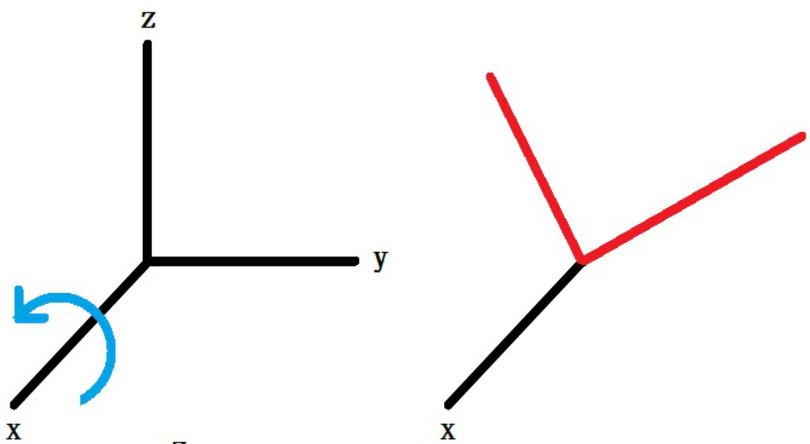
3D cases

The previous 2D case can be now treated as a 3D case



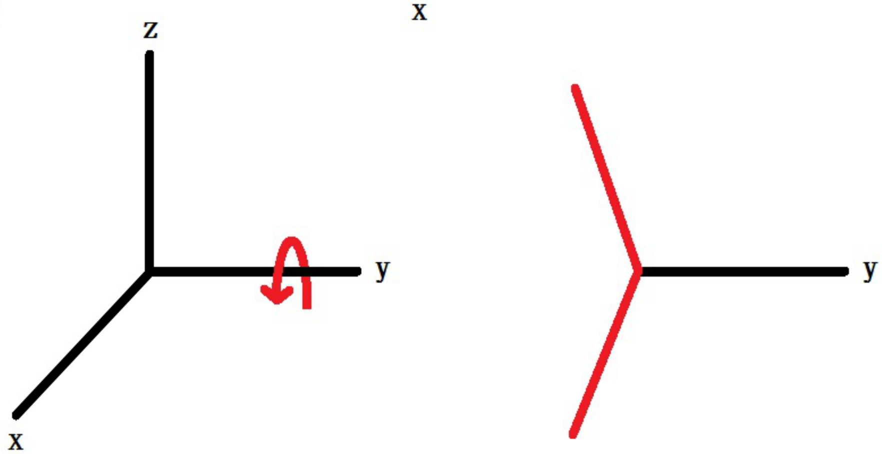
In 3D case, the general rotation matrix for frame i w.r.t to frame j is

$$R_i^j = \begin{bmatrix} x_i \cdot x_j & y_i \cdot x_j & z_i \cdot x_j \\ x_i \cdot y_j & y_i \cdot y_j & z_i \cdot y_j \\ x_i \cdot z_j & y_i \cdot z_j & z_i \cdot z_j \end{bmatrix}$$



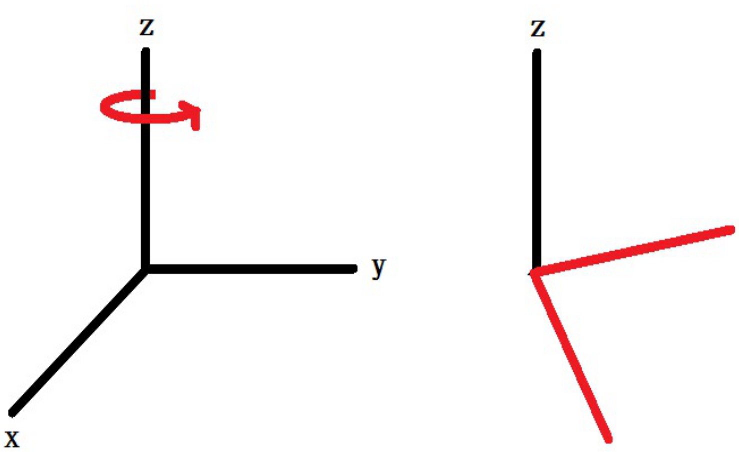
Rotation along x axis :

$$R_1^0(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Rotation along y axis

$$R_1^0(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



Rotation along z axis :

$$R_1^0(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why rotation along y axis seems a little bit different in the sign

The rotation is defined as positive if it is rotated anti-clockwise along that axis. In the case that rotation along the x,z axis, it fulfill this requirement. But for the case that rotate along y axis, the angle is clockwise, so a negative sign is added into the angle.

Multiple Rotation

Rotation about the current frame
For point p in base frame, after the first rotation , it rotate again

$$\text{What is } p^2 \text{ w.r.t frame 0 ?}$$

Since p^2 and p^1 is related by

$$p^1 = R_2^1 p^2$$

And p^1 and p^0 is related by

$$p^0 = R_1^2 p^1$$

Thus

$$p^0 = R_1^0 R_2^1 p^2$$

And thus for

$$p^0 = R_2^0 p^2$$

$$R_2^0 = R_1^0 R_2^1$$

That is, if

$$\text{Frame0} \xrightarrow{R_1^0} \text{Frame1} \xrightarrow{R_2^1} ... \xrightarrow{R_n^{n-1}} \text{Frame n}$$

$$p^0 = R_1^0 R_2^1 ... R_{n-1}^{n-2} R_n^{n-1} p^n$$

And

$$R_n^0 = R_1^0 R_2^1 ... R_{n-1}^{n-2} R_n^{n-1}$$

$$-END-$$