Homogenous matrix

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Transformation of unit vector w.r.t different frame

A point p is described by 2 frame : $o_0x_0y_0$ and $o_1x_1y_1$. Where frame 1 is frame 0 translated by a distance d (a vector), and then rotated by a rotation matrix R. Question: what is the relationship between p^1 and p^0 ?

Answer:

$$p^0 = Rp^1 + d$$

where

$$R = R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \qquad d = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

A dimention 4 matrix can be used to store the information of both translation and rotation. Such matrix is called homogenous matrix H:

$$H = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 & d_x \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 & d_y \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus rotation matrix is thus

$$R = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 & 0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 & 0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$R(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R(z,\theta) = \begin{bmatrix} \cos\theta 1 & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And translation matrix is thus

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$T\left(x,d\right) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T(y,d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; T(z,d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore

$$p^0 = Rp^1 + d \qquad \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

can be written as

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 & d_x \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 & d_y \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}$$

Properties of the homogenous matrix

$$H = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 & d_x \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 & d_y \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} & R & \end{pmatrix}_{3\times3} & \begin{pmatrix} d \end{pmatrix}_{3\times1} \\ 0_{1\times3} & 1_{1\times1} \end{bmatrix} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

Consecutive multiplications of H matrix do not change the form

$$\begin{bmatrix} R_1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & d_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & d_2 + R_1d_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R' & d' \\ 0 & 1 \end{bmatrix}$$

Since R is an orthogonal matrix : $R^T R^{-1} = 1$

$$\det\left(R^T R^{-1}\right) = 1$$

$$\det\left(R^T\right)\det\left(R^{-1}\right) = 1$$

The inverse

$$\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}^{-1} = \left(\det \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}\right)^{-1} \operatorname{cof} \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \frac{\left(M_{ij} \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}\right)^T}{\det \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}} = \frac{1}{||R||} \begin{bmatrix} 1 & -d \\ 0 & R \end{bmatrix} = \frac{R}{||R||}$$

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