Denavit–Hartenberg Convention

May 15, 2013

1 The DH Convention

The homogenous transformation include 6 degree of freedom (3 translational, 3 rotational)

When using Denavit-Hartenberg Convention, the degree of freedom is reduced to 4 (2 rotational 2 translational)

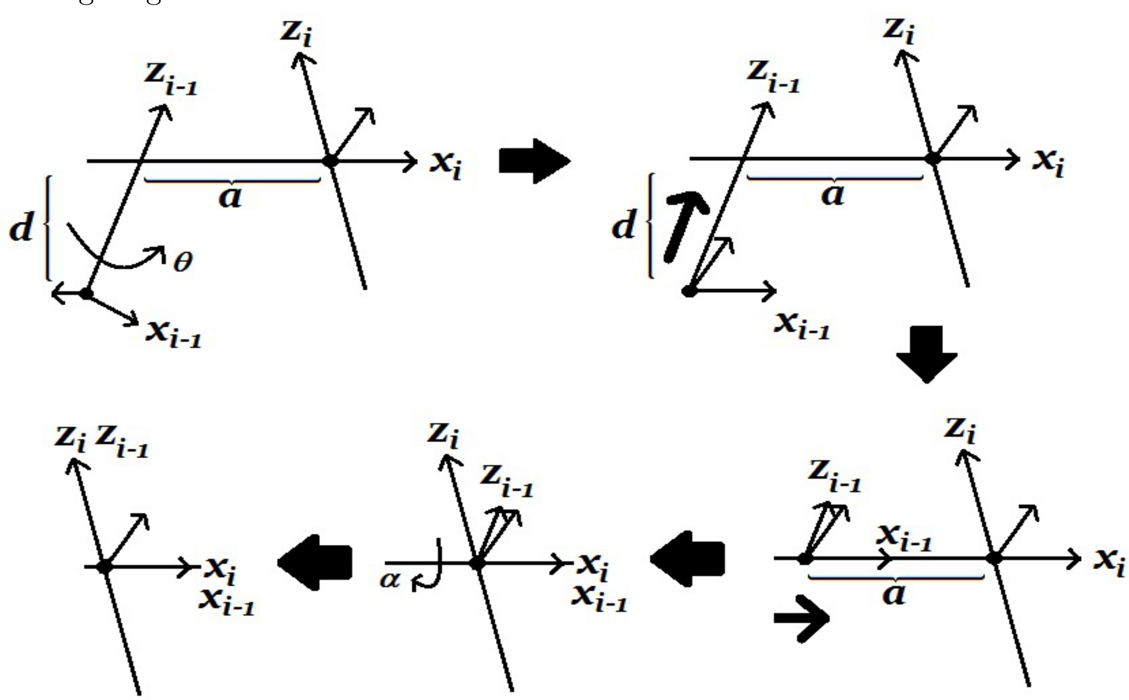
The homogenous transformation can be expressed as 4 basic transformations

$$H_i^j = \mathrm{Rot}(z, \theta_i)\mathrm{Trans}(z, d_i)\mathrm{Trans}(x, a_i)\mathrm{Rot}(x, \alpha_i)$$

Where

$$\begin{cases} \theta_i\text{: joint angle} \\ d_i \text{ : link offset} \\ a_i \text{ : link length} \\ \alpha_i\text{: link twist} \end{cases} \qquad \begin{cases} \mathrm{Rot}(z, \theta_i) \text{ : rotate along z-axis} \\ \mathrm{Trans}(z, d_i) \text{ : translate along z-axis} \\ \mathrm{Trans}(x, a_i) \text{ : translate along x-axis} \\ \mathrm{Rot}(z, \theta_i) \text{ :rotate along x-axis} \end{cases}$$

The following diagram shows that is DH-convention

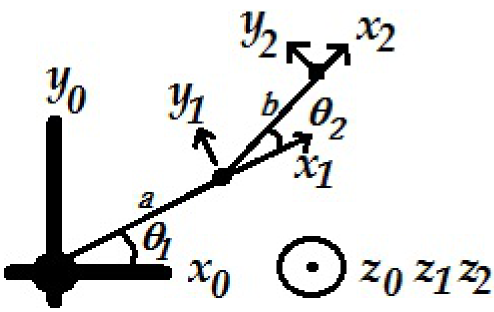


$$H_i^j = \mathrm{Rot}(z, \theta_i)\mathrm{Trans}(z, d_i)\mathrm{Trans}(x, a_i)\mathrm{Rot}(x, \alpha_i)$$

Denote $c_\theta = \cos\theta$, $s_\theta = \sin\theta$

$$H_i^j = \begin{pmatrix} c_\theta & -s_\theta & & \\ s_\theta & c_\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & d_i \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & a_i \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & c_\theta & -s_\theta & \\ & s_\theta & c_\theta & \\ & & & 1 \end{pmatrix}$$

2 Examples : Double Pendulum



Link	θ	d	a	α
1	θ_1	0	a	0
2	θ_2	0	b	0

Explanation
$\theta = \angle$rotate along z
d =distance along z , as the coordinates are on same plane so $d_1 = d_2 = 0$
a =distance along x
$\alpha = \angle$rotate along x , as $z_0 \shortparallel z_1 \shortparallel z_2$ (all pointing out of paper) , so $\alpha_1 = \alpha_2 = 0$

$$H_0^2 = H_0^1 H_1^2 = \begin{pmatrix} c_{\theta 1} & -s_{\theta 1} & & \\ s_{\theta 1} & c_{\theta 1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & a \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} c_{\theta 2} & -s_{\theta 2} & & \\ s_{\theta 2} & c_{\theta 2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & b \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}$$

$$H_0^2 = \begin{pmatrix} c_{\theta 1} & -s_{\theta 1} & & \\ s_{\theta 1} & c_{\theta 1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & a \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c_{\theta 2} & -s_{\theta 2} & & \\ s_{\theta 2} & c_{\theta 2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & b \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{\theta 1} & -s_{\theta 1} & & c_{\theta 1} a \\ s_{\theta 1} & c_{\theta 1} & & s_{\theta 1} a \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c_{\theta 2} & -s_{\theta 2} & & c_{\theta 2} b \\ s_{\theta 2} & c_{\theta 2} & & s_{\theta 2} b \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{\theta 1} c_{\theta 2} - s_{\theta 1} s_{\theta 2} & -c_{\theta 1} s_{\theta 1} - c_{\theta 2} s_{\theta 2} & & c_{\theta 1} c_{\theta 2} b - s_{\theta 1} s_{\theta 2} b + c_{\theta 1} a \\ s_{\theta 1} c_{\theta 2} + s_{\theta 2} c_{\theta 1} & -s_{\theta 1} s_{\theta 2} + c_{\theta 1} c_{\theta 2} & & s_{\theta 1} c_{\theta 2} b + c_{\theta 1} s_{\theta 2} b + s_{\theta 1} a \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

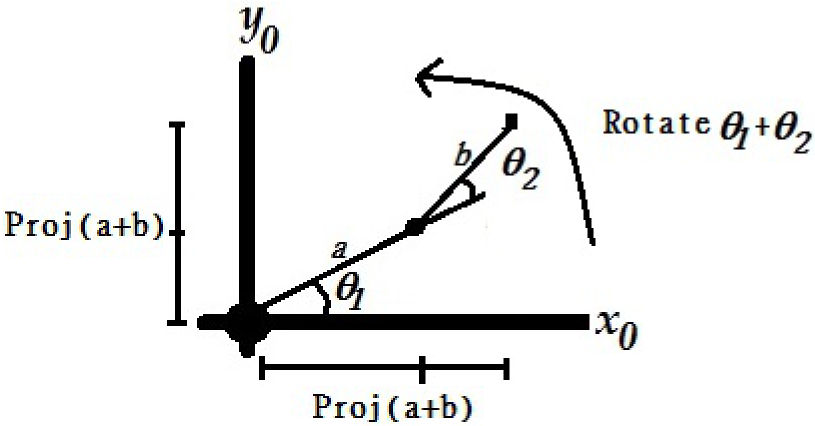
$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & & \cos(\theta_1 + \theta_2)\, b + \cos(\theta_1)\, a \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & & \sin(\theta_1 + \theta_2)\, b + \sin(\theta_1)\, a \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Which means

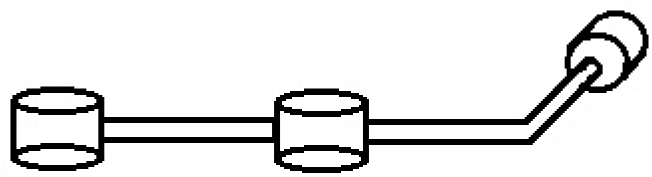
$$= \left(\begin{pmatrix} & & \\ & \mathrm{Rot}\left(z, \theta_1 + \theta_2\right) & \\ & & \end{pmatrix} \begin{pmatrix} \mathrm{proj}_{x_0} a + \mathrm{proj}_{x_0} b \\ \mathrm{proj}_{y_0} a + \mathrm{proj}_{y_0} b \\ 0 \end{pmatrix} \atop \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad\qquad 1 \right)$$

$\begin{pmatrix} \mathrm{proj}_{x_0} a + \mathrm{proj}_{x_0} b \\ \mathrm{proj}_{y_0} a + \mathrm{proj}_{y_0} b \\ 0 \end{pmatrix}$ is the position of end-point w.r.t. base frame $O_0X_0Y_0Z_0$

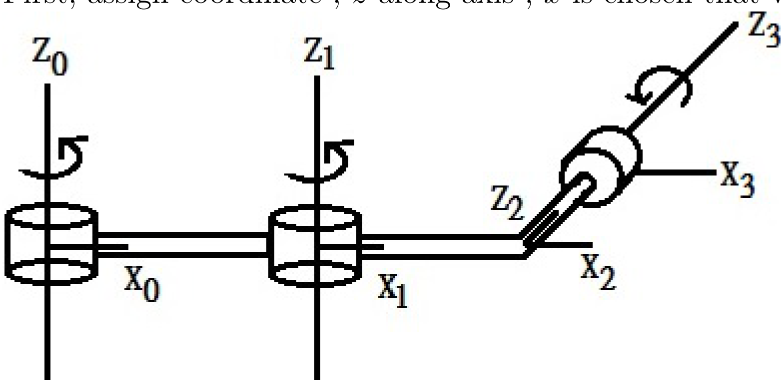
$\begin{pmatrix} \\ \mathrm{Rot}\left(z, \theta_1 + \theta_2\right) \\ \\ \end{pmatrix}$ is the orientation of end-point w.r.t. base frame $O_0X_0Y_0Z_0$



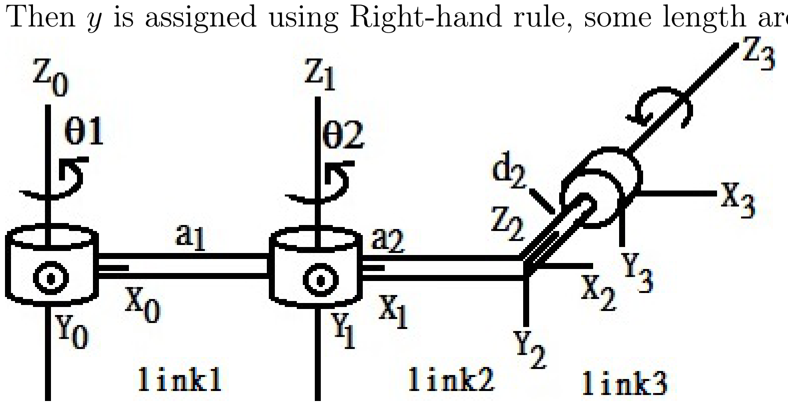
3 Example II. RRR



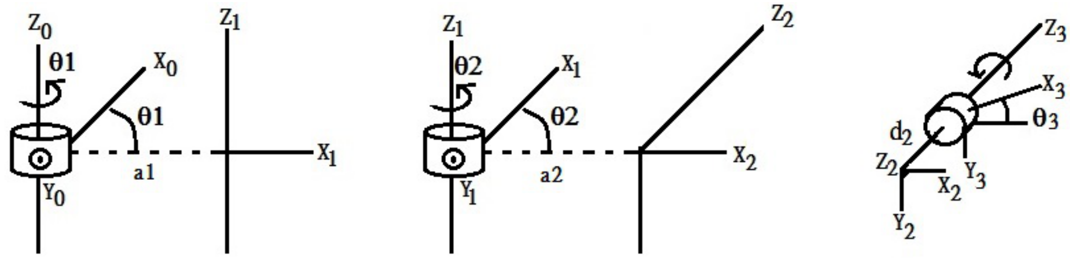
First, assign coordinate , z along axis , x is chosen that will simplify the problem



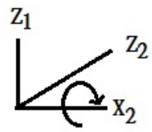
Then y is assigned using Right-hand rule, some length are labeled



Link	θ	d	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	-90^o
3	θ_3	d_2	0	0



	θ1	θ2	θ3
Angle between x about z	θ1	θ2	θ3
Distance between origin along z	no such difference	no such difference	d2
Distance between origin along x	a1	a2	no such difference
Angle between z about x	zero, because z0//z1	-90	zero



Thus the homogenous transformation matrix $H_0^3 = H_0^1 H_1^2 H_2^3$

$$
\begin{pmatrix} c_{\theta1} & -s_{\theta1} & & \\ s_{\theta1} & c_{\theta1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & a_1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
$$

$$
= \times
\begin{pmatrix} c_{\theta2} & -s_{\theta2} & & \\ s_{\theta2} & c_{\theta2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & a_2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}
$$

$$
\times
\begin{pmatrix} c_{\theta3} & -s_{\theta3} & & \\ s_{\theta3} & c_{\theta3} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & d_2 \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
$$

$$
=
\begin{pmatrix} c_{\theta1} & -s_{\theta1} & & \\ s_{\theta1} & c_{\theta1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & a_1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} c_{\theta2} & -s_{\theta2} & & \\ s_{\theta2} & c_{\theta2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
$$

$$
\times
\begin{pmatrix} 1 & & & a_2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} c_{\theta3} & -s_{\theta3} & & \\ s_{\theta3} & c_{\theta3} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & d_2 \\ & & & 1 \end{pmatrix}
$$

$$
=
\begin{pmatrix} c_{\theta1} & -s_{\theta1} & & c_{\theta1}a_1 \\ s_{\theta1} & c_{\theta1} & & s_{\theta1}a_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} c_{\theta2} & -s_{\theta2} & & c_{\theta2}a_2 \\ s_{\theta2} & c_{\theta2} & & s_{\theta2}a_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} c_{\theta3} & -s_{\theta3} & & \\ s_{\theta3} & c_{\theta3} & & \\ & & 1 & d_2 \\ & & & 1 \end{pmatrix}
$$

$$
=
\begin{pmatrix} c_{\theta1+\theta2} & -s_{\theta1+\theta2} & & c_{\theta1+\theta2}a_2 + c_{\theta1}a_1 \\ s_{\theta1+\theta2} & -c_{\theta1+\theta2} & & s_{\theta1+\theta2}a_2 + s_{\theta1}a_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} c_{\theta3} & -s_{\theta3} & & \\ & & -1 & -d_2 \\ s_{\theta3} & c_{\theta3} & & \\ & & & 1 \end{pmatrix}
$$

$$
=
\begin{pmatrix} c_{\theta1+\theta2}c_{\theta3} & -c_{\theta1+\theta2}s_{\theta3} & s_{\theta1+\theta2} & s_{\theta1+\theta2}d_2 + c_{\theta1+\theta2}a_2 + c_{\theta1}a_1 \\ s_{\theta1+\theta2}c_{\theta3} & -s_{\theta1+\theta2}s_{\theta3} & c_{\theta1+\theta2} & c_{\theta1+\theta2}d_2 + s_{\theta1+\theta2}a_2 + s_{\theta1}a_1 \\ s_{\theta3} & & 1 & \\ & & & 1 \end{pmatrix}
$$

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