Trajectory Generation via Spline Cubic Polynomial

March 26, 2013

A cubic polynomial is in the form

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

The velocity and acceleration functions are thus

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\ddot{q}(t) = 2a_2 + 6a_3 t$$

Trajectory without obstacles

To solve for the unknown coefficients, we need to have the boundary condition $\begin{cases} q(t_s) = A \\ \dot{q}(t_s) = B \\ q(t_s) = C \\ \dot{q}(t_f) = D \end{cases}$

These are the user required performance of the movement of the robotic arm

$$\begin{cases} a_0 + a_1 t_s + a_2 t_s^2 + a_3 t_s^3 = A \\ a_1 + 2a_2 t_s + 3a_3 t_s^2 = B \\ a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 = C \\ a_1 + 2a_2 t_f + 3a_3 t_f^2 = D \end{cases} \iff \begin{bmatrix} 1 & t_s & t_s^2 & t_s^3 \\ 0 & 1 & 2t_s & 3t_s^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Usually the start time t_s is zero

$$\begin{cases} a_0 = A \\ a_1 = B \\ a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 = C \\ a_1 + 2a_2 t_f + 3a_3 t_f^2 = D \end{cases} \iff \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

The matrix is not singular

$$\det \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ t_f & t_f^2 & t_f^3 \\ 1 & 2t_f & 3t_f^2 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{bmatrix} = t_f^4 \neq 0$$

After solving

$$\begin{cases} a_0 = A \\ a_1 = B \\ a_2 = \dfrac{3\left(C - A\right) - t_f\left(2B + D\right)}{t_f^2} \\ a_3 = \dfrac{2\left(A - C\right) + t_f\left(B + D\right)}{t_f^3} \end{cases}$$

And thus the equation (displacement, velocity and acceleration) are

$$\begin{cases} q(t) = A + Bt + \dfrac{3(C-A) - t_f(2B+D)}{t_f^2}t^2 + \dfrac{2(A-C) + t_f(B+D)}{t_f^3}t^3 \\ \dot{q}(t) = B + \dfrac{6(C-A) - 2t_f(2B+D)}{t_f^2}t + \dfrac{6(A-C) + 3t_f(B+D)}{t_f^3}t^2 \\ \ddot{q}(t) = \dfrac{6(C-A) - 2t_f(2B+D)}{t_f^2} + \dfrac{12(A-C) + 6t_f(B+D)}{t_f^3}t \end{cases}$$

Since usually the initial and final velocity is zero (stop at both end), thus in these cases $B = D = 0$, and the eqaution reduce to

$$\begin{cases} q(t) = A + \dfrac{3(C-A)}{t_f^2}t^2 + \dfrac{2(A-C)}{t_f^3}t^3 \\ \dot{q}(t) = \dfrac{6(C-A)}{t_f^2}t + \dfrac{6(A-C)}{t_f^3}t^2 \\ \ddot{q}(t) = \dfrac{6(C-A)}{t_f^2} + \dfrac{12(A-C)}{t_f^3}t \end{cases}$$

Trajectory with obstacles

When the trajectory need to avoid some obstacles, that means the trajectory should not pass through some points.

To make it simple, that means the trajectory should pass through some points that can avoid hitting the obstacles.

Suppose in addition to the upper part, now the trajectory has to pass throguh points p_1 at time t_p

	starting point p_s	p_1	end point p_e
displacement	known	given	known
velocity	zero	?	zero

How to determine the unknown velocity ?

The most simple idea is to choose the velocity (they are free variable) that the acceleration is continuous in these point.

As acceleration means force, and a drag force that suddenly appear is not good for robotic arm.

Thus first , split the path from one to 2 path

Original path is $p_s \to p_e$, now $p_s \to p_1 \to p_e$

Since there are 2 path, that means there is 2 displacement equations $\begin{cases} a_0 + a_1t + a_2t^2 + a_3t^3 & t \in [t_s, t_p] \\ b_0 + b_1t + b_2t^2 + b_3t^3 & t \in [t_p, t_e] \end{cases}$

Let the velocity at time t_p be v
Solve the 2 equations, and equal the acceleration equations to solve for the unknowns.

−END−