General 2^{nd} Order System

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1. General Form of 2^{nd}order T.F.
2. Rewive of related Mathematics
3. Gain in frequency & time domain : $G_{\text{II}}(s) \leftrightarrow g_{\text{II}}(t)$
4. Different ζ , the dimensionless damping ratio

1 General 2^{nd}order T.F.

$$G_{\text{II}}(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$X_O(s) = \left(\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) X_I(s) = \left(\frac{K}{\left(\dfrac{s}{\omega_n}\right)^2 + 2\zeta\dfrac{s}{\omega_n} + 1}\right) X_I(s)$$

$$\begin{cases} K = \text{Proportion Gain} \\ \zeta = \text{Relative Damping Ratio, dimensionless} \\ \omega_n = \text{Undamped natural resonant frequency} \\ \text{Characteristic Equation : } s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \end{cases}$$

For unit impulse input , $X_I(s) = U(s)$

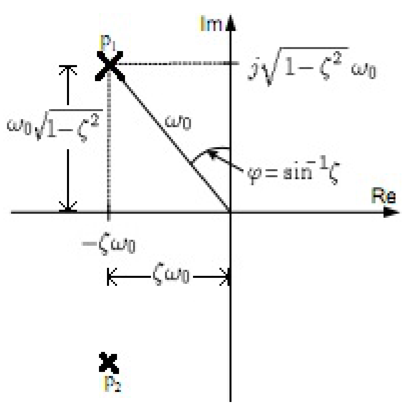
$$X_O(s) = \left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) U(s)$$

Where $U(s) = \mathcal{L}\{u(t)\} = \dfrac{U}{s}$, U : amplitude

2 Review of related Mathematics

- Characteristic Equation : $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
- Fundamental Theorem of Algebra : $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \iff (s - p_1)(s - p_2) = 0$
- Pole : $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- Sum of root : $p_1 + p_2 = -2\zeta\omega_n$
- Product of root : $p_1 p_2 = \omega_n^2$

- Difference of root : $p_1 - p_2 = \sqrt{(p_1+p_2)^2 - 4p_1p_2} = \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2} = 2\omega_n\sqrt{\zeta^2-1}$
- $\dfrac{p_1p_2}{p_1-p_2} = \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}$
- $\mathbb{C}$: $\forall z \in \mathbb{C}$, it can be expressed as : $z = R + jI = |z|e^{j\theta}$
- $|z| = \sqrt{R^2+I^2} \qquad \theta = \tan^{-1}\dfrac{I}{R}$
- For $z = \dfrac{1}{R+jI} \qquad \Longrightarrow \begin{cases} z = \dfrac{R-jI}{R^2+I^2} \\ |z| = \sqrt{Re^2+Im^2} = \dfrac{1}{\sqrt{R^2+I^2}} \\ \angle z = \tan^{-1}\dfrac{-I}{R} = -\tan^{-1}\dfrac{I}{R} \end{cases}$
- For $\sqrt{x^2-1}$, if $x<1$, $\sqrt{x^2-1} = j\sqrt{1-x^2} \in \mathbb{C}$
- Euler Formula $\sin\theta = \dfrac{e^{j\theta}-e^{-j\theta}}{2j}$
- Hyperbolic : $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2}$
- If $\zeta \geqslant 1$, pole$\in \mathbb{R}$, unequal , $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$
- If $\zeta = 1$, pole$\in \mathbb{R}$, eq. $p = -\omega_n$
- If $0<\zeta<1$, pole$\in \mathbb{C}$-conjugate , $p_{1,2} = \underbrace{-\zeta\omega_n}_{\text{Re}} \pm j \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\text{Im}}$



- Laplace Transform Pair : $\mathcal{L}\{e^{+at}f(t)\} = F(s-a)$

3 Gain in Frequency & Time domain $G_{\mathbf{II}}(s)$ & $g_{\mathbf{II}}(t)$

$$G_{\text{II}}(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{Kp_1p_2}{(s-p_1)(s-p_2)}$$

Partial Fraction Decomposition via Heaviside Cover Up Method

$$G_{\text{II}}(s) = \frac{\frac{Kp_1p_2}{p_1-p_2}}{s-p_1} + \frac{\frac{Kp_1p_2}{p_2-p_1}}{s-p_2} = \frac{Kp_1p_2}{p_1-p_2}\left(\frac{1}{s-p_1} - \frac{1}{s-p_2}\right)$$

By $\dfrac{Kp_1p_2}{p_1-p_2}=\dfrac{k\omega_n}{2\sqrt{\zeta^2-1}}$

$$G_{\text{II}}(s)=\frac{k\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{1}{s-p_1}-\frac{1}{s-p_2}\right)$$

Take $\mathcal{L}^{-1}$, $G_{\text{II}}(s)\leftrightarrow g_{\text{II}}(t)$

$$g_{\text{II}}(t)=\mathcal{L}^{-1}\left\{G_{\text{II}}(s)\right\}=\frac{k\omega_n}{2\sqrt{\zeta^2-1}}\left\{e^{p_1t}-e^{p_2t}\right\}$$

Put $p_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$

$$\begin{aligned}g_{\text{II}}(t)&=\frac{k\omega_n}{2\sqrt{\zeta^2-1}}\left\{e^{\left(-\zeta\omega_n+\sqrt{\zeta^2-1}\omega_n\right)t}-e^{\left(-\zeta\omega_n-\sqrt{\zeta^2-1}\omega_n\right)t}\right\}\\&=\frac{k\omega_n}{2\sqrt{\zeta^2-1}}\left\{e^{-\zeta\omega_nt}e^{\sqrt{\zeta^2-1}\omega_nt}-e^{-\zeta\omega_nt}e^{-\sqrt{\zeta^2-1}\omega_nt}\right\}\end{aligned}$$

Rearrange, the **General 2^{nd}order Gain in time domain is**

$$g_{\text{II}}(t)=\frac{k\omega_n}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_nt}\left\{\frac{e^{\sqrt{\zeta^2-1}\omega_nt}-e^{-\sqrt{\zeta^2-1}\omega_nt}}{2}\right\}$$

4 Differnet ζ , the damping ratio

4.1 $\zeta>1$

- Overdamping
- $G_{\text{II}}(s)=\dfrac{K\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}=\dfrac{Kp_1p_2}{(s-p_1)(s-p_2)}$
- Af $\zeta>1$, $\sqrt{\zeta^2-1}\in\mathbb{R}$, pole$\in\mathbb{R}$, $p_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$

$$g_{\text{II}}(t)=\frac{K\omega_n}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_nt}\left\{\frac{e^{\sqrt{\zeta^2-1}\omega_nt}-e^{-\sqrt{\zeta^2-1}\omega_nt}}{2}\right\}$$

$$g_{\text{II}}(t)=\frac{K\omega_n}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_nt}\sinh\left(\sqrt{\zeta^2-1}\right)\omega_nt$$

4.2 $\zeta=1$

- $G_{\text{II}}(s)=\dfrac{K\omega_n^2}{s^2+2\omega_ns+\omega_n^2}=\dfrac{K\omega_n^2}{(s+\omega_n)^2}=\dfrac{Kp^2}{(s-p)^2}$
- Pole $\in\mathbb{R}$, eq, $p=-\omega_n$
- As $\zeta=1$, $\sqrt{\zeta^2-1}=0$, can not use the method above

$$g_{II}(t)=\mathcal{L}^{-1}\left\{\frac{K\omega_n^2}{(s-p)^2}\right\}=K\omega_n^2\mathcal{L}\left\{(s-p)^{-2}\right\}=K\omega_n^2t\cdot e^{pt}=K\omega_n^2te^{-\omega_nt}$$

4.3 $0 < \zeta < 1$

- $G_{\text{II}}(s) = \dfrac{K\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \dfrac{Kp_1p_2}{(s-p)(s-p_2)}$
- Pole $\in \mathbb{C}$-conjugate, $p_{1,2} = \underbrace{-\zeta\omega_n}_{\text{Re}} \pm j \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\text{Im}}$
- As $\zeta < 1$, $\sqrt{\zeta^2 - 1} = j\sqrt{1-\zeta^2} \in \mathbb{C}$

$$g_{\text{II}}(t) = \frac{K\omega_n}{\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t} \left\{ \frac{e^{\sqrt{\zeta^2-1}\omega_n t} - e^{-\sqrt{\zeta^2-1}\omega_n t}}{2} \right\}$$

$$g_{\text{II}}(t) = \frac{K\omega_n}{j\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left\{ \frac{e^{j\sqrt{1-\zeta^2}\omega_n t} - e^{-j\sqrt{1-\zeta^2}\omega_n t}}{2} \right\}$$

$$g_{\text{II}}(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\omega_n t$$

4.4 $\zeta = 0$

- No Damping, as ζ represent damping ratio
- $G_{\text{II}}(s) = \dfrac{K\omega_n^2}{s^2 + \omega_n^2} = K\omega\dfrac{\omega}{s^2 + \omega_n^2}$
- $p = \pm\omega_n \in \mathbb{R}$, uneq.
- $g(t)$ can be obtain by taking $\mathcal{L}^{-1}$, $g_{\text{II}}(t) = \mathcal{L}^{-1}\{G_{\text{II}}(s)\} = \mathcal{L}^{-1}\{K\omega\dfrac{\omega}{s^2+\omega_n^2}\} = K\omega_n \sin\omega_n t$
- Or using $g_{\text{II}}(t) = \dfrac{K\omega_n}{\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t} \left\{ \dfrac{e^{\sqrt{\zeta^2-1}\omega_n t} - e^{-\sqrt{\zeta^2-1}\omega_n t}}{2} \right\}$, as $\zeta = 0$, $\sqrt{\zeta^2 - 1} = j$

$$g_{\text{II}}(t) = \frac{K\omega_n}{j} e^0 \left\{ \frac{e^{j\omega_n t} - e^{-j\omega_n t}}{2} \right\}$$

$$g_{\text{II}}(t) = K\omega_n \sin\omega_n t$$

4.5 $\zeta < 0$

- Unstable
- $G_{\text{II}}(s) = \dfrac{K\omega_n^2}{s^2 - 2|\zeta|\omega_n s + \omega_n^2}$
- $e^{-\zeta\omega_n t} = e^{|\zeta|\omega_n t}$ Diverge as $t \to \infty$

$$g_{\text{II}}(t) = \frac{K\omega_n}{\sqrt{\zeta^2-1}} e^{|\zeta|\omega_n t} \left\{ \frac{e^{\sqrt{\zeta^2-1}\omega_n t} - e^{-\sqrt{\zeta^2-1}\omega_n t}}{2} \right\}$$

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