Frequency Response of Sinusoidal Input

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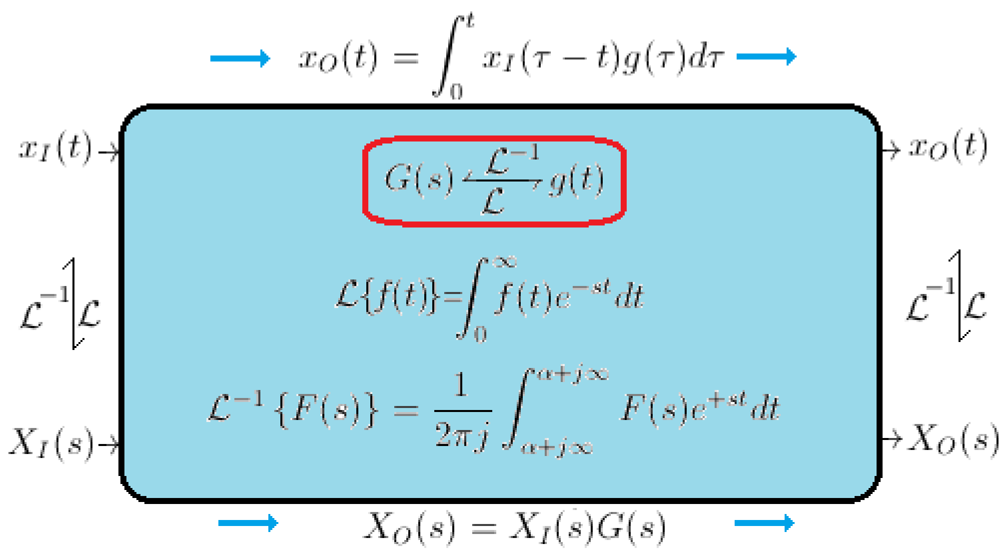


1 Introduction

For sinusoidal input,

$$x_I(t) = A \sin(\omega t)$$

The output is the convolution of input and impulse response

$$x_I(t) = \int_0^t x(\tau - t) g(\tau) d\tau = x(t) * g(t)$$

This integration may be very messy

Find the output by $x_O(t) = \mathcal{L}^{-1}\{X_O(s)\}$ where $X_O(s) = X_I(s)G(s)$

2 The Laplace Transfrom of sinusoidal input

$$X_I(s) = \frac{A\omega}{s^2 + \omega^2}$$

This part is the derivation this laplace transform , this part can be skipped.

$$X_I(s) = \mathcal{L}\{x_I(t)\} = \int_0^\infty A \sin(\omega t) e^{-st} dt$$

Method 1. Direct Integration using by Parts (Skip)

Method 2. Apply Euler Formula

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \sin\theta = Im\left[e^{j\theta}\right]$$

$$\mathcal{L}\left\{e^{j\omega t}\right\} = \int_0^\infty e^{j\omega t}e^{-st}dt = \int_0^\infty e^{-(s-j\omega)t}dt = \left[\frac{e^{-(s-j\omega)t}}{s-j\omega}\right]_0^\infty = \frac{1}{s-j\omega}$$

Rationalize

$$= \frac{s+j\omega}{s^2+\omega^2} = \underbrace{\frac{s}{s^2+\omega^2}}_{Re} + j\underbrace{\frac{\omega}{s^2+\omega^2}}_{Im}$$

As $\mathcal{L}$ is an *linear operator* ,

$$\mathcal{L}\left\{e^{j\omega t}\right\} = \mathcal{L}\left\{\cos\omega t + j\sin\omega t\right\} = \mathcal{L}\left\{\cos\omega t\right\} + j\mathcal{L}\left\{\sin\omega t\right\}$$

So

$$\mathcal{L}\left\{\sin\omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

So

$$\mathcal{L}\left\{x_I(t)\right\} = \mathcal{L}\left\{A\sin\omega t\right\} = \frac{A\omega}{s^2+\omega^2}$$

3 For sinusoidal input, output is also sinusoidal

$$X_O(s) = \mathcal{L}\left\{x_I(t)\right\}\mathcal{L}\left\{g(t)\right\} = X_I(s)G(s) = \frac{A\omega}{s^2+\omega^2}G(s)$$

The general transfer function ,$G(s)$

$$G(s) = \frac{\sum_{i=1}^{i=m} b_i s^i}{\sum_{i=1}^{i=n} a_i s^i} = \frac{b_m s^m + ... + b_1 s + b_0}{a_n s^n + ... + a_1 s + a_0}$$

By Fundamental Theorem of Algebra, it can be expressed as

$$G(s) = k\frac{\prod_{i=1}^{i=m}(s-z_i)}{\prod_{i=1}^{i=n}(s-p_i)} = \frac{k(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

After cancellig common factor, the denominator and numerator become coprime. Then express the $G(s)$ into partial fraction.

$$G(s) = \sum_{i=1}^{n}\frac{\lambda_i}{s+\alpha_i}$$

So the output in frequency domain is

$$X_O(s) = X_I(s)G(s) = \frac{A\omega}{s^2+\omega^2}G(s)$$

Break it into partial fraction (Using Heaviside Cover Up Method)

$$X_O(s) = \frac{\left(\frac{A\omega G(s)}{s-j\omega}\right)_{s=-j\omega}}{s+j\omega} + \frac{\left(\frac{A\omega G(s)}{s+j\omega}\right)_{s=j\omega}}{s-j\omega} + \sum_{i=1}^{n}\frac{\lambda_i}{s+\alpha_i}$$

$$= \frac{AG(-j\omega)}{-2j\,(s+j\omega)} + \frac{AG(+j\omega)}{2j\,(s-j\omega)} + \sum_{i=1}^{n} \frac{\lambda_i}{s+\alpha_i}$$

$$= j\frac{AG(-j\omega)}{2\,(s+j\omega)} - j\frac{AG(+j\omega)}{2\,(s-j\omega)} + \sum_{i=1}^{n} \frac{\lambda_i}{s+\alpha_i}$$

Let $G(\pm j\omega) = R(\omega) \pm jI(\omega)$

$$= \frac{A\,[jR(\omega)+I(\omega)]}{2\,(s+j\omega)} - \frac{A\,[jR(\omega)-I(\omega)]}{2\,(s-j\omega)} + \sum_{i=1}^{n} \frac{\lambda_i}{s+\alpha_i}$$

$$X_O(s) = \frac{AR(\omega)}{2} j \left[\frac{1}{s+j\omega} - \frac{1}{s-j\omega}\right] + \frac{AI(\omega)}{2}\left[\frac{1}{s+j\omega} + \frac{1}{s-j\omega}\right] + \sum_{i=1}^{n} \frac{\lambda_i}{s+\alpha_i}$$

Take $\mathcal{L}^{-1}$

$$x_O(t) = AR(\omega) \underbrace{\left(\frac{j}{2}\right)\left(e^{-j\omega t} - e^{+j\omega t}\right)}_{\sin\omega t} + AI(\omega) \underbrace{\left(\frac{e^{-j\omega t}+e^{j\omega t}}{2}\right)}_{\cos\omega t} + \sum_{i=1}^{n} \lambda_i e^{-\alpha_i t}$$

$$x_O(t) = \underbrace{AR(\omega)\sin\omega t + AI(\omega)\cos\omega t} + \sum_{i=1}^{n} \lambda_i e^{-\alpha_i t}$$

$$x_O(t) = A\sqrt{[R(\omega)]^2 + [I(\omega)]^2}\,\sin\left(\omega t + \tan^{-1}\frac{I(\omega)}{R(\omega)}\right) + \sum_{i=1}^{n} \lambda_i e^{-\alpha_i t}$$

Recall, $G(\pm j\omega) = R(\omega) \pm I(\omega)$

- $|G(j\omega)| = \sqrt{[R(\omega)]^2 + [I(\omega)]^2}$
- $\angle G(+j\omega) = \tan^{-1}\frac{I(\omega)}{R(\omega)} \quad \angle G(-j\omega) = \tan^{-1}\frac{-I(\omega)}{R(\omega)}$

The total response is thus

$$x_O(t) = A\Big|G(j\omega)\Big|\sin\left(\omega t + \angle G(j\omega)\right) + \sum_{i=1}^{n} \lambda_i e^{-\alpha_i t}$$

If for all the α , $Re\,(\alpha) > 0$, then

$$\sum_{i=1}^{n} \lambda_i e^{-\alpha_i t} = \sum_{i=1}^{n} \lambda_i e^{-(Re(\alpha_i)+jIm(\alpha_i))t} = \sum_{i=1}^{n} \lambda_i e^{-Re(\alpha_i)t} e^{-jIm(\alpha_i)t}$$

When $t \to \infty$,

$$\lim_{t\to\infty} \sum_{i=1}^{n} \lambda_i e^{-\alpha_i t} = \lim_{t\to\infty} \sum_{i=1}^{n} \lambda_i e^{-Re(\alpha_i)t} e^{-jIm(\alpha_i)t} = 0$$

So the *Steady state response* of the system with **Sinusoidal Input is also Sinusoidal with changes in magnitude and phase**

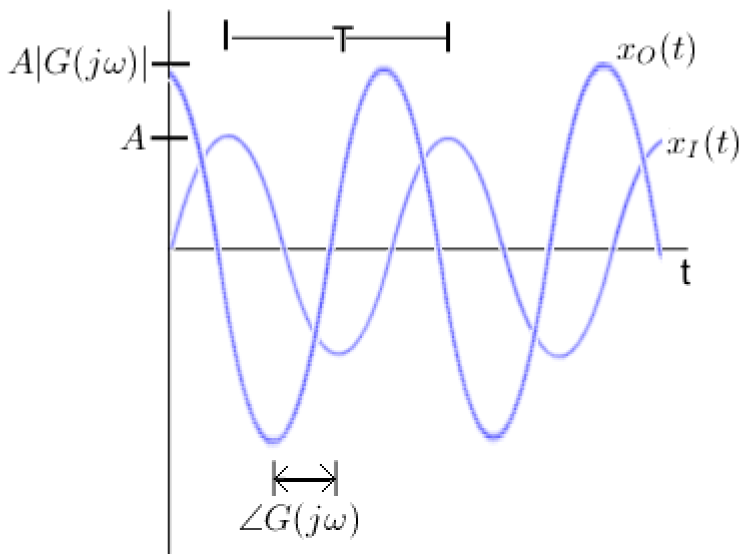
$$x_I(t)\ ,\ x_O(t) \in Sinusoidal$$

$$x_I(t) = A\sin{(\omega t)}$$

$$x_O(t) = A|G(j\omega)|\sin{(\omega t + \angle G(j\omega))}$$

$$|x_O(t)| = |x_I(t)||G(j\omega)|$$

$$\angle x_O(t) = \angle x_I(t) + \angle G(j\omega)$$



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