SFG and Mason’s Rule : A revision

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Review

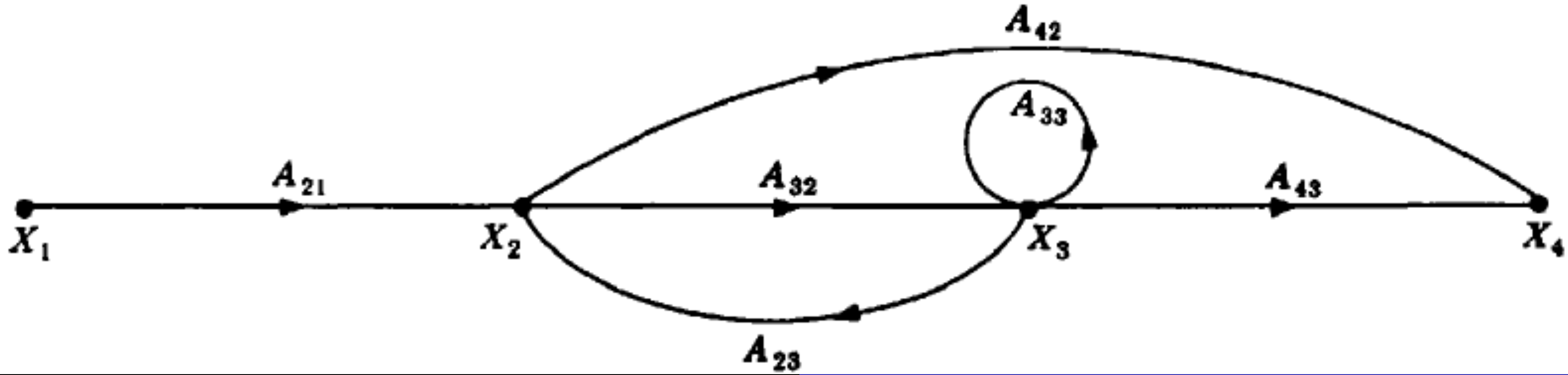
- SFG: **S**ignal-**F**low **G**raph
- SFG is a *directed graph*
- SFG is used to model signal flow in a system
- SFG can be used to derive the transfer function of the system by *Mason's Rule / Mason's Gain Formula*.

SFG terminologies

- Node
- Edge
- Gain
- Input / Sources
- Output / Sinks
- Path
- Path gain
- Forward path
- Forward path gain
- Loop
- Self loop
- Loop Gain
- Non-touching loop

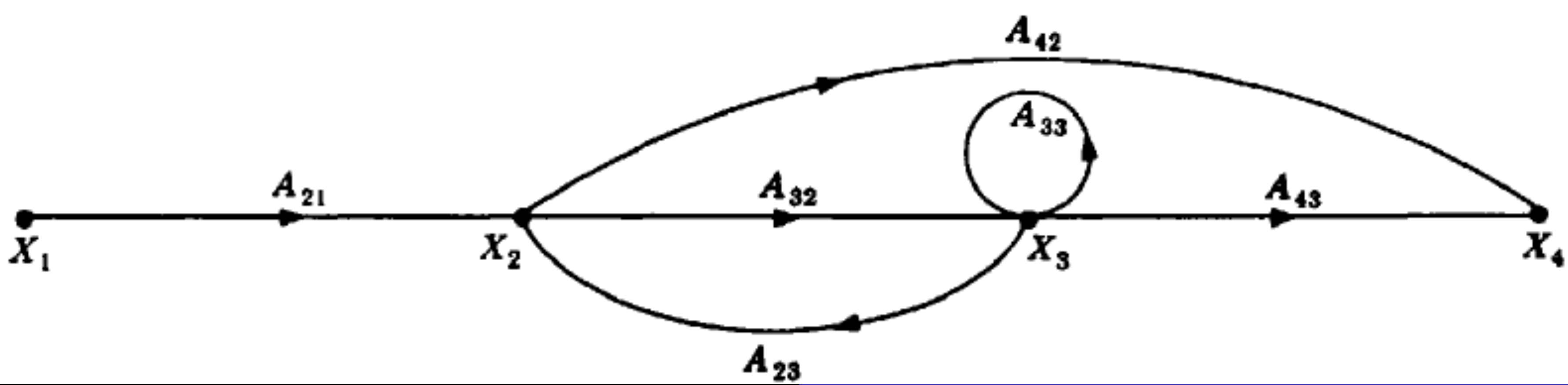
SFG terminologies

- Node: variables. e. g. X_1, X_2, X_3, X_4
- Edge: directed branches. e. g. $X_1 X_2$
- Gain: transmission of that branch. e. g. A_{21}
- Input/Sources: nodes with out-going branches only e. g. X_1
- Output/Sinks: nodes with incoming branches only e. g. X_4



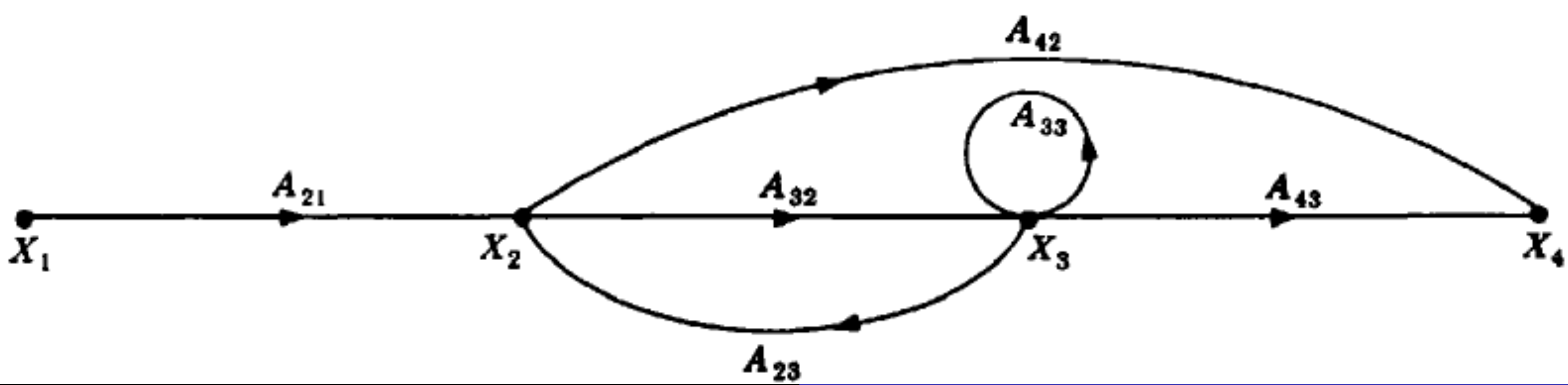
SFG terminologies

- Path: successive branches without repeated nodes, e. g. $X_1X_2X_4$
- Path gain: the gain of the path
- Forward path: path from input to output, e. g. $X_1X_2X_3X_4$ or $X_1X_2X_4$
- Forward path gain: gain of forward path



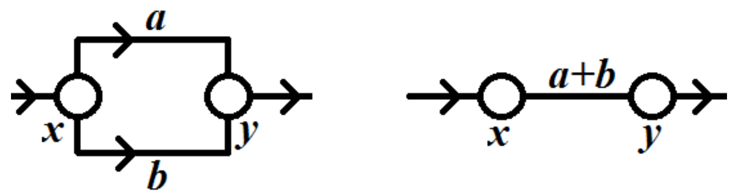
SFG terminologies

- Loop / Feedback path: a closed path which originates and terminates on the same node, e. g. $X_2X_3X_2$
- Self loop: loop with only one branch, e.g. X_3X_3
- Loop gain: gain along the loop
- Non-touching loop: two loops are non-touching if they do not share any nodes nor branches

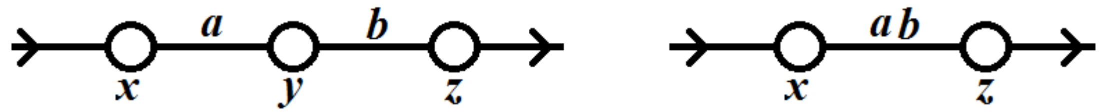


Some SFG simplifications

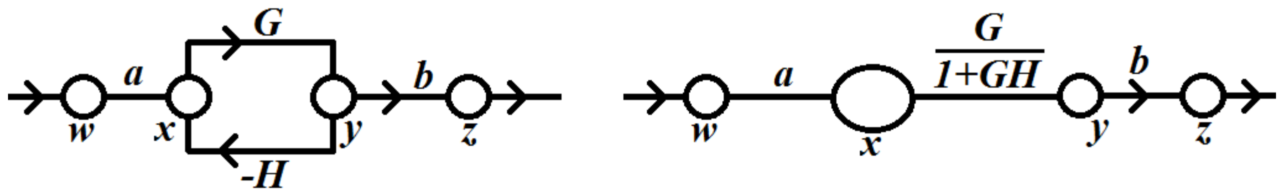
- Branch in parallel:



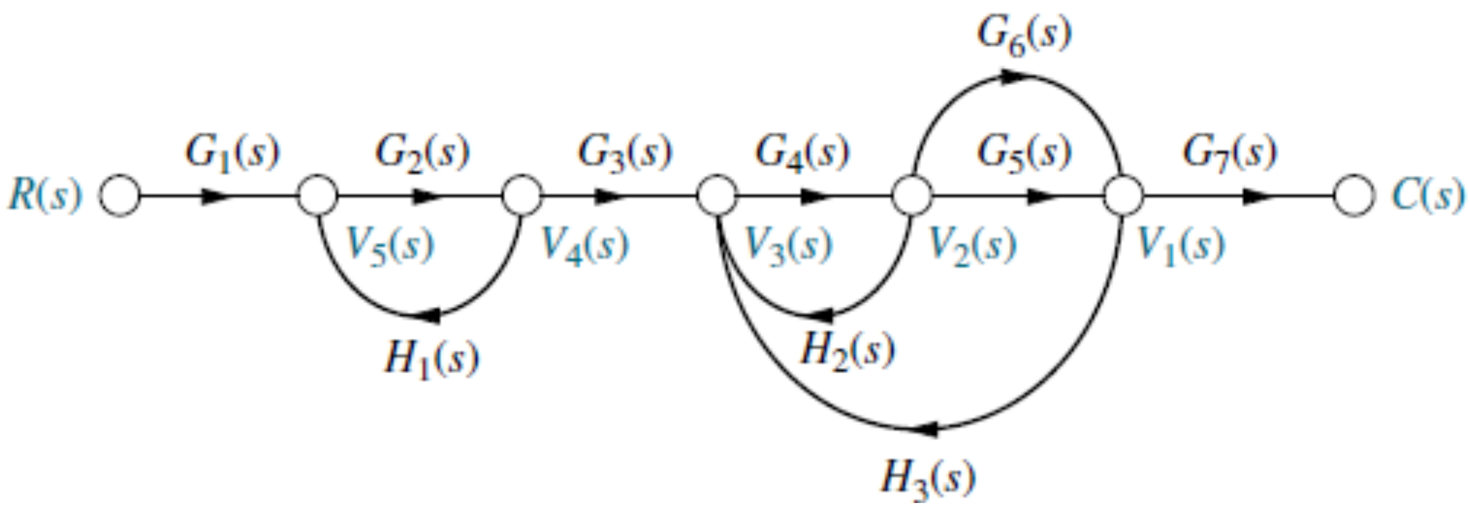
- Branch in series:



- Isolated loop:

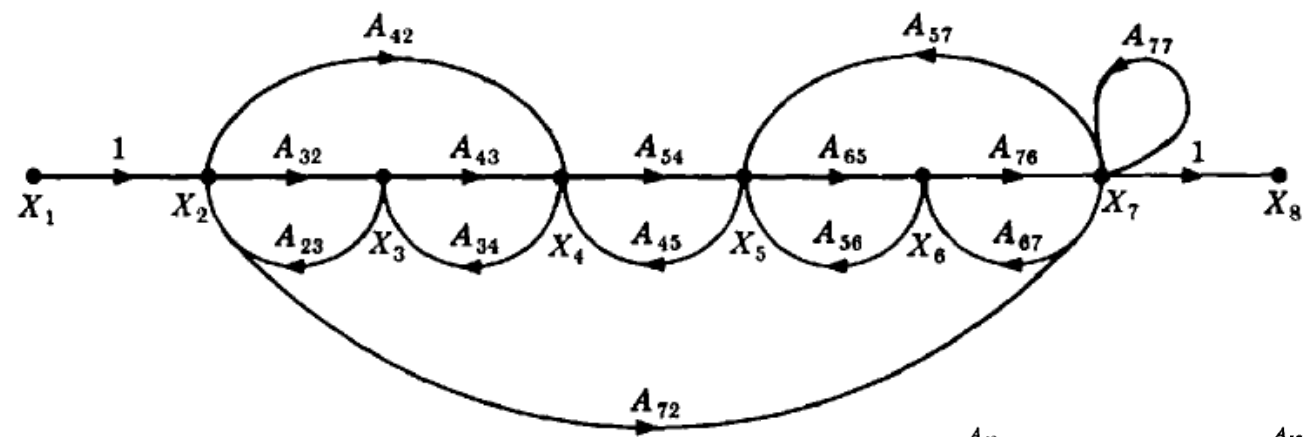


SFG Example 1

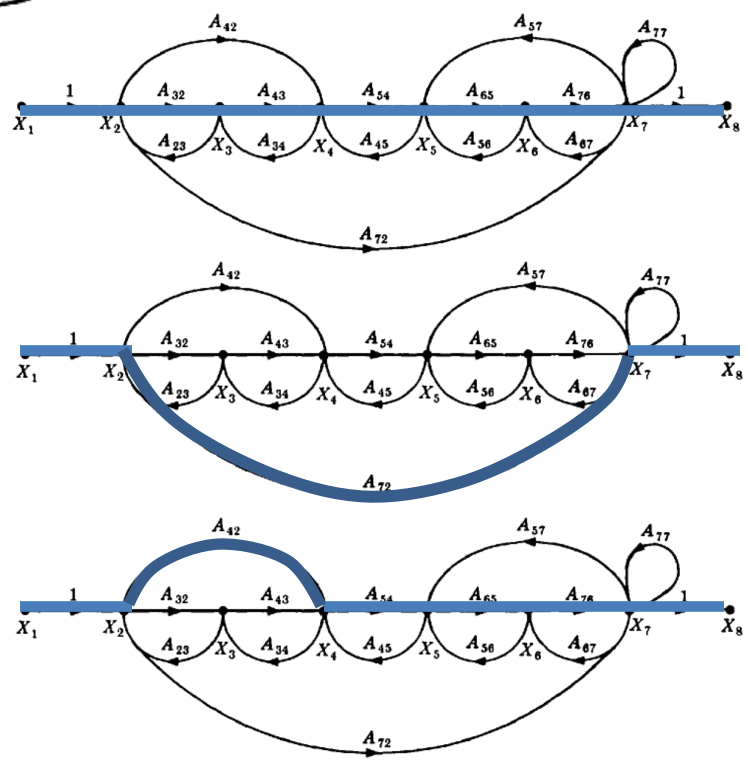


- Two forward path: $G_1G_2G_3G_4G_5G_7$ & $G_1G_2G_3G_4G_6G_7$
- Four loops: G_2H_1, G_4H_2, $G_4G_5H_3$, $G_4G_6H_3$
- Non-touching loops: G_2H_1 & G_4H_2, G_2H_1 & $G_4G_5H_3$, G_2H_1 & $G_4G_6H_3$

SFG Example 2



- Three forward path:



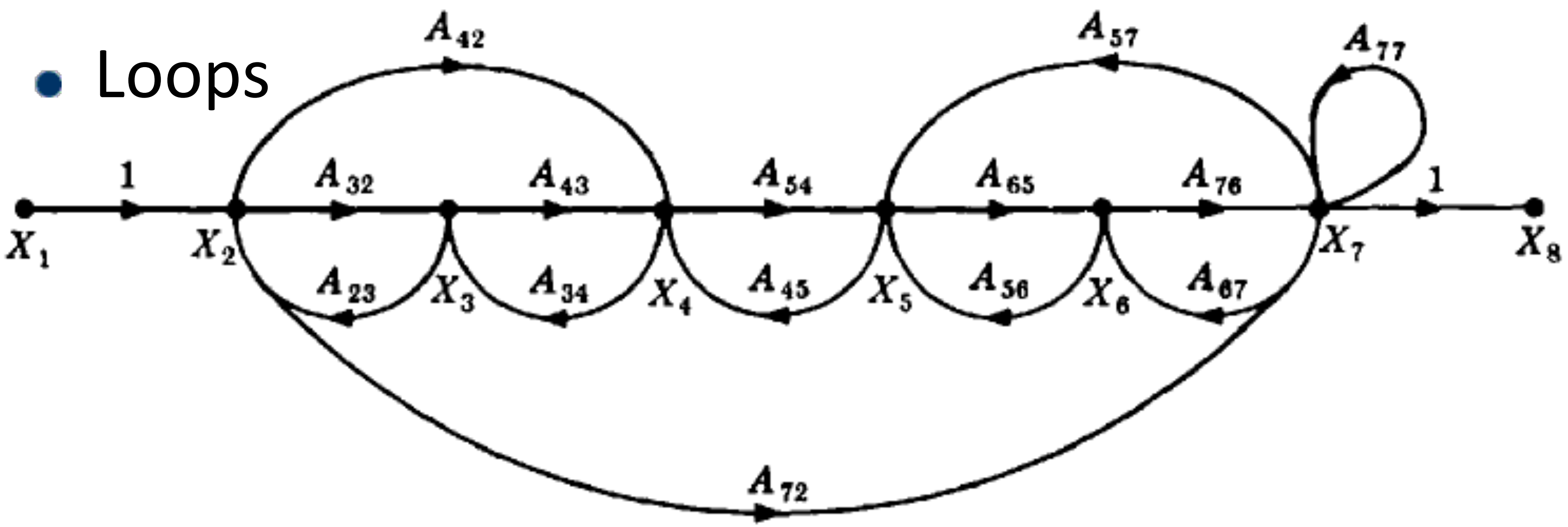
$A_{32}A_{43}A_{54}A_{65}A_{76}$

A_{72}

$A_{42}A_{54}A_{65}A_{74}$

SFG Example 2

- Loops



- $A_{32}A_{33}$, $A_{43}A_{34}$, $A_{54}A_{45}$, $A_{65}A_{56}$, $A_{76}A_{67}$, A_{77}
- $A_{42}A_{34}A_{23}$, $A_{65}A_{76}A_{57}$
- * $A_{65}A_{76}A_{67}A_{56}$ is not a loop since X_6 is repeated on the path!
- $A_{72}A_{57}A_{45}\,A_{34}A_{23}$
- $A_{72}A_{67}A_{56}A_{45}\,A_{34}A_{23}$

Mason’s Rule

- How to derive transfer function: block diagram (BD) reduction or signal flow graph reduction.
- BD approach requires successive application of fundamental relationships in order to derive transfer function.
- SFG just applies one formula – Mason’s Rule

Mason's Rule

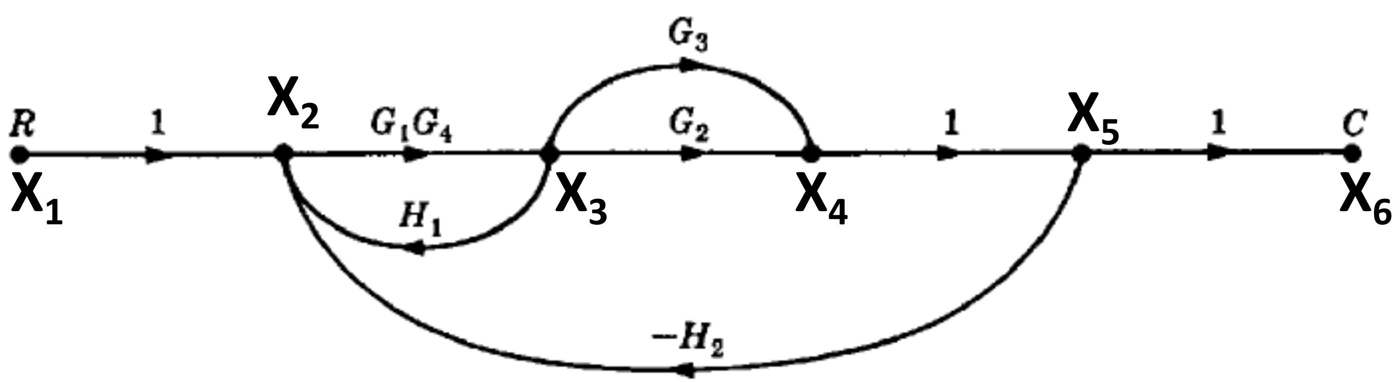
- Equation $$TF(s) = \frac{1}{\Delta}\left(\sum_{i=1}^{\#\text{forward path}} P_i\Delta_i\right)$$
  - P_i = the i^{th} forward-path gain.
  - Δ = Determinant of the system
  - Δ_i = Determinant of the i^{th} forward path
- Δ = 1 - (sum of all individual loop gains)

+ (sum of products of gains of all 2 loops that do not touch each other)

– (sum of products of gains of all possible three loops that do not touch each other) + …

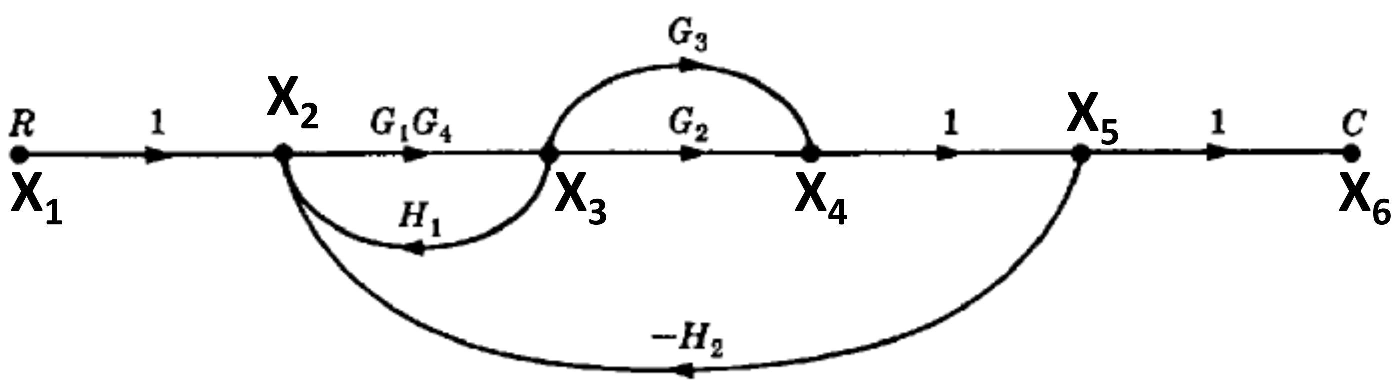
- Δ_i = Δ for part of SFG that does not touch i-th forward path
- Δ_i = 1 if no non-touching loops to the i-th path, or if taking out i-th path breaks all the loops

Example 1



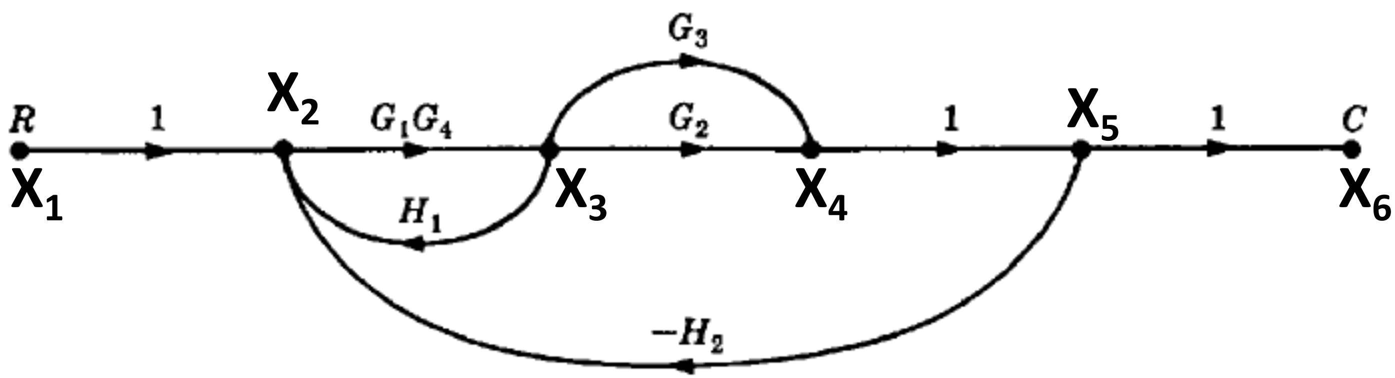
- Input: X_1
- Output: X_6
- **Two** Forward paths: paths along $X_1X_2X_3X_4X_5X_6$
- **Two** Forward path gains: $P_1 = G_1G_2G_4$ and $P_2 = G_1G_3G_4$
- Loops: $X_2X_3X_2$ and **two** loops along $X_2X_3X_4X_5X_2$
- Loops gains: $L_1 = G_1G_4H_1$, $L_2 = -G_1G_2G_4H_2$ and $L_3 = -G_1G_3G_4H_2$

Example 1



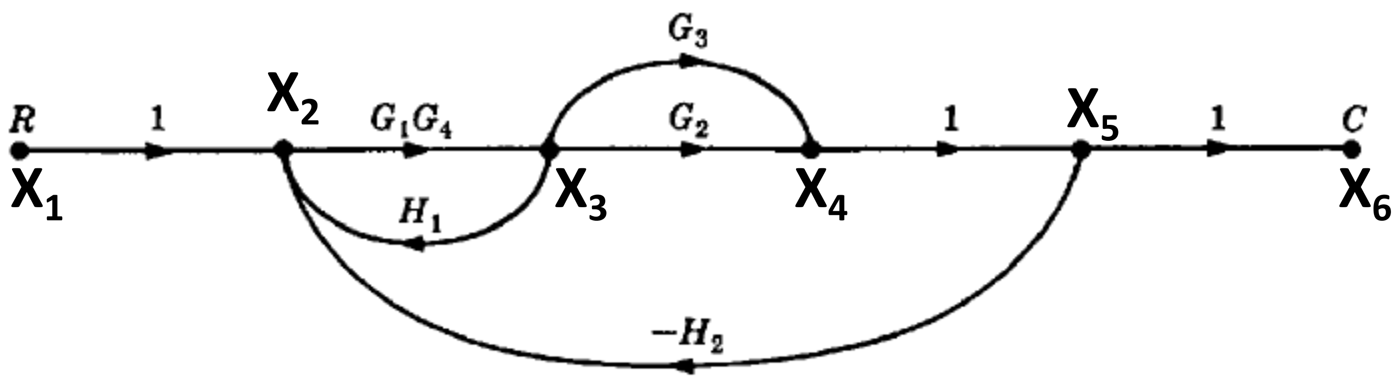
- $\Delta =$ 1 - (sum of all individual loop gains) + (sum of products of gains of all 2 loops that do not touch each other) – (sum of products of gains of all possible three loops that do not touch each other)
- Since no non-touching loop, thus
- $\Delta =$ 1 - (sum of all individual loop gains)
- $\Delta = 1 - (L_1 + L_2 + L_3)$

Example 1



- $\Delta_i = \Delta$ for part of SFG that does not touch i-th forward path ($\Delta_i = 1$ if no non-touching loops to the i-th path)
- Since no non-touching loop, thus $\Delta_1 = \Delta_2 = 1$

Example 1



$$TF = \frac{1}{\Delta}\left(\sum_{i=1}^{\#\text{forward path}} P_i\Delta_i\right)$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_1)$$

$$= \frac{1}{1-(L_1+L_2+L_3)}(P_1+P_2)$$

$$= \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1\ G_3G_4H_2}$$

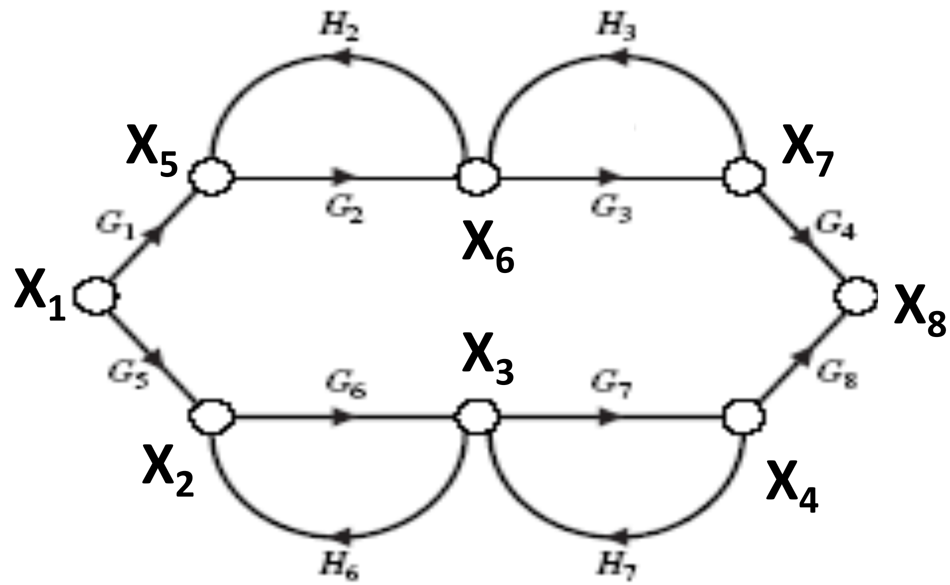
Example 2

- Input X_1, Output X_8
- **Two** Forward paths:

  $X_1X_2X_3X_4X_8$ and $X_1X_5X_6X_7X_8$

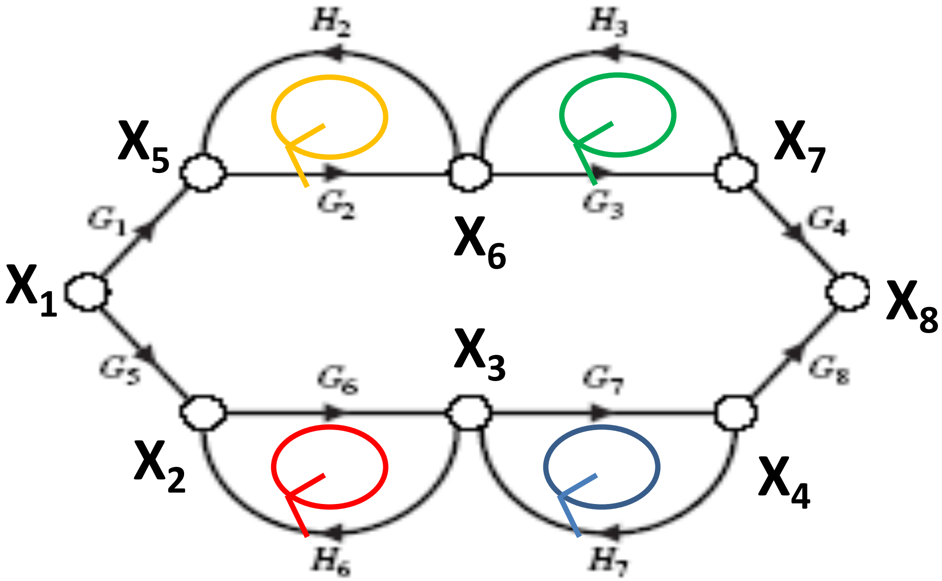
- **Two** Forward paths gain:

$$P_1 = G_1G_6G_7G_8 \quad P_2 = G_1G_2G_3G_4$$



Example 2

- Loops: $X_2X_3X_2$, $X_3X_4X_3$, $X_5X_6X_5$, $X_6X_7X_6$



- Individual Loop gains:

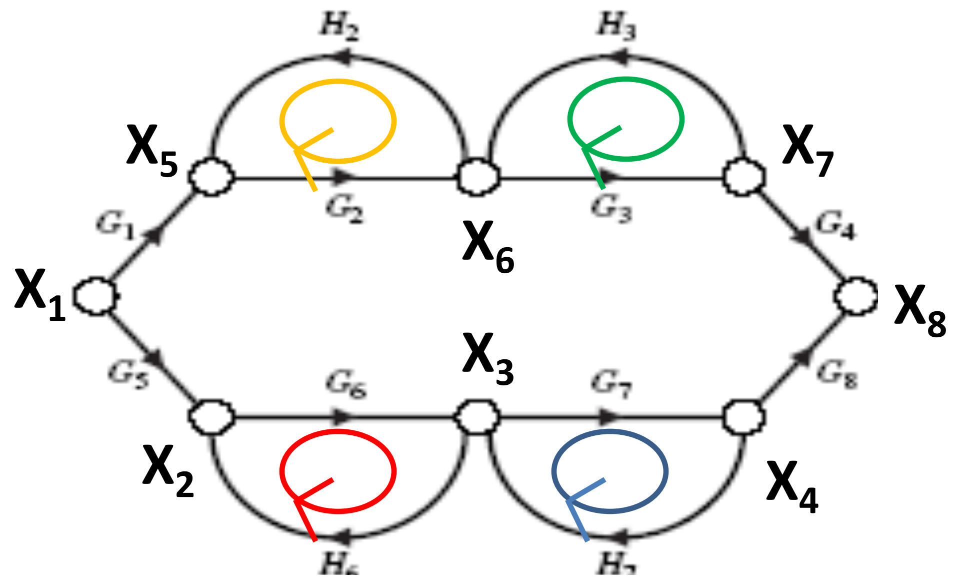
$L_{11} = G_6H_6 \quad L_{12} = G_7H_7,$

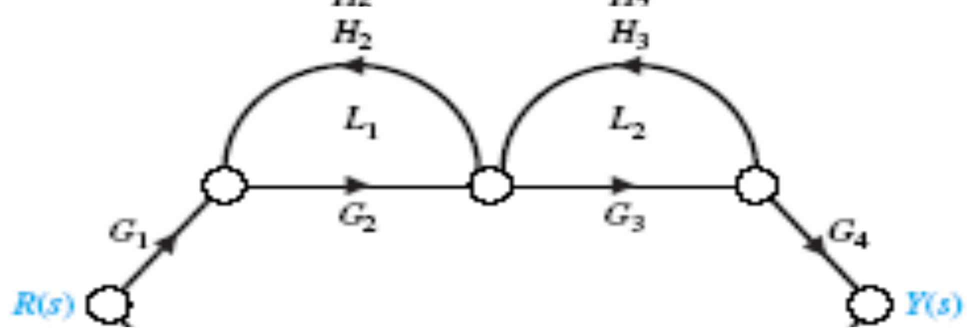
$L_{13} = G_2H_2 \quad L_{14} = G_3H_3$

- Two non-touching loops:
- $L_{21} = L_{11}L_{13} \quad L_{22} = L_{11}L_{14} \quad L_{23} = L_{12}L_{13} \quad L_{24} = L_{12}L_{14}$
- Three non-touching loops: no
- $\Delta = 1 - (L_{11} + L_{12} + L_{13} + L_{14}) + (L_{21} + L_{22} + L_{23} + L_{24})$

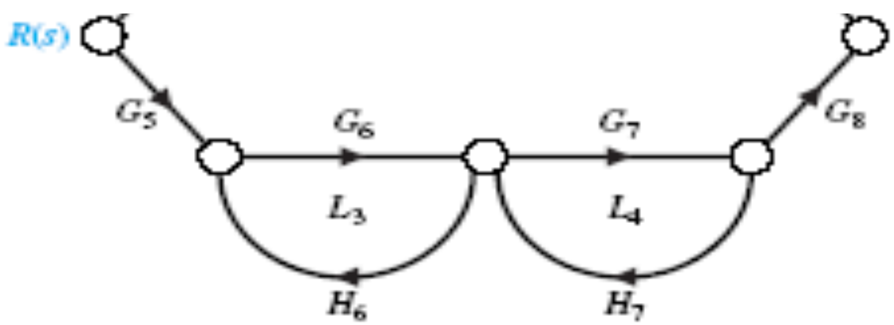
Example 2

- Δ_i = Δ for part of SFG that does not touch i-th forward path (Δ_i = 1 if no non-touching loops to the i-th path.)





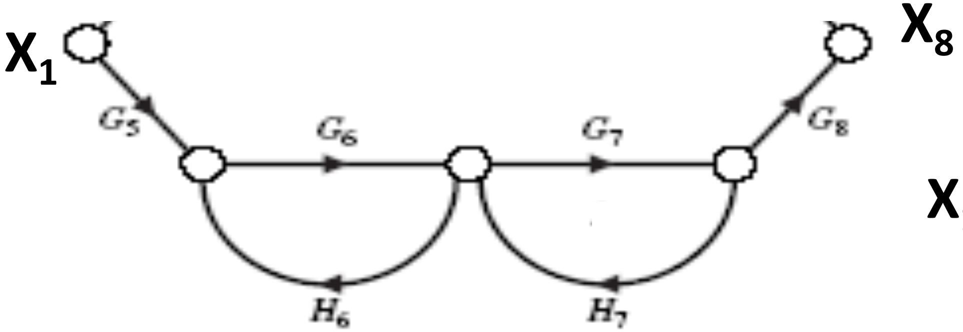
$\Delta_2 = \Delta$ for part of SFG that does not touch 2nd forward path

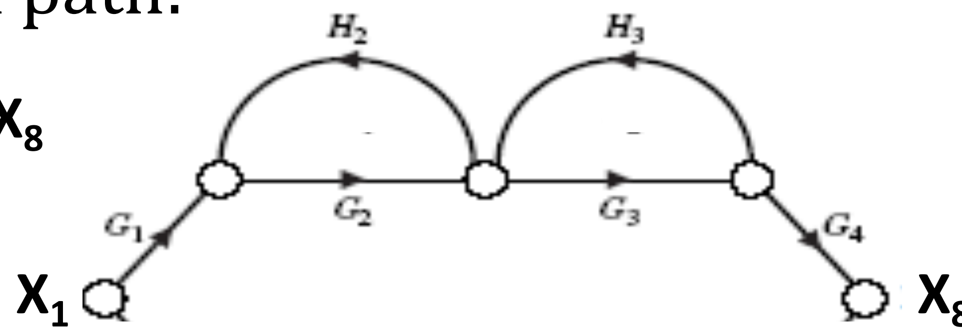


$\Delta_1 = \Delta$ for part of SFG that does not touch 1st forward path

Example 2

SFGs without the i-th forward path:



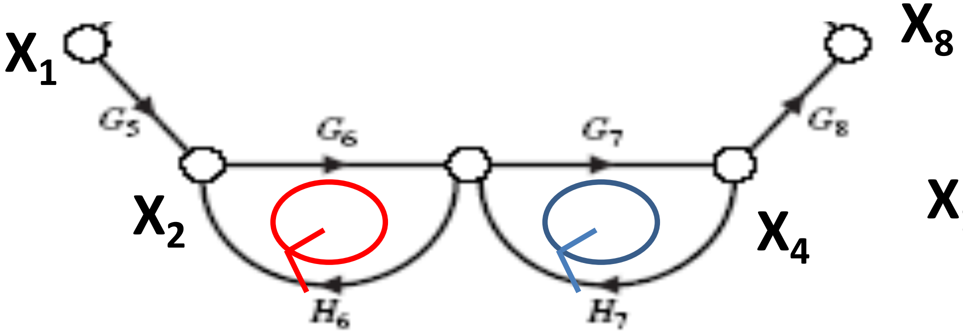


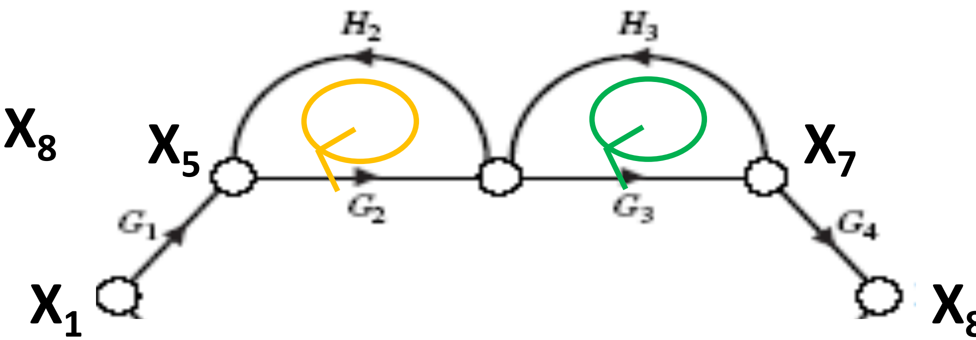
For these SFGs, compute their Δ as:

Δ = 1 - (sum of all individual loop gains) + (sum of products of gains of all 2 loops that do not touch each other) – (sum of products of gains of all possible three loops that do not touch each other)

Example 2

Both SFGs do not have two/more non-touching loops



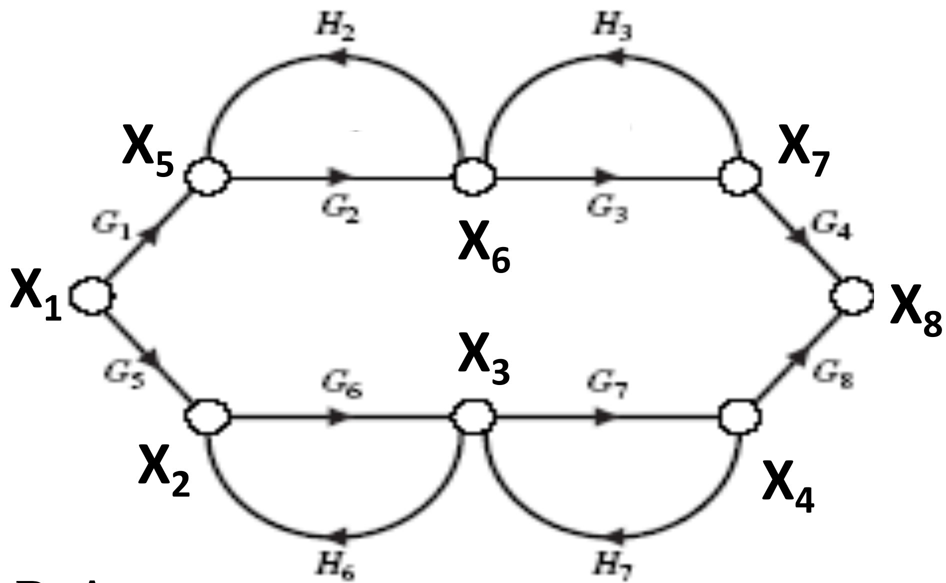


$$\Delta_2 = 1 - (L_{11} + L_{12})$$
$$= 1 - G_6 H_6 - G_7 H_7$$

$$\Delta_1 = 1 - (L_{13} + L_{14})$$
$$= 1 - G_2 H_2 - G_3 H_3$$

Example 2

$$\text{TF} = \frac{1}{\Delta}\left(\sum_{i=1}^{\#\text{forward path}} P_i \Delta_i\right)$$



$$= \frac{P_1\Delta_1 + P_2\Delta_2}{1 - (L_{11} + L_{12} + L_{13} + L_{14}) + (L_{21} + L_{22} + L_{23} + L_{24})}$$

$$= \frac{G_1G_6G_7G_8(1 - G_2H_2 - G_3H_3) + G_1G_2G_3G_4(1 - G_6H_6 - G_7H_7)}{1 - (G_2H_2 + G_3H_3 + G_6H_6 + G_7H_7) + (G_2H_2G_6H_6 + G_2H_2G_7H_7 + G_3H_3G_6H_6 + G_3H_3G_7H_7)}$$

END