Routh - Hurwitz Stability Criterion

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Reference DiStefano III, R. Stubberud, Ivan J. Williams *Feedback and Control Systems*

1 Review of Related Mathematics

1.1 Limit, Convergence, Bound

Diverge and Converge , Limit For exponential function e^x

$$\begin{cases} \text{Converge : } \lim_{x \to +\infty} e^x = 0 \iff x \text{ is positive} \\ \text{Diverge : } \lim_{x \to +\infty} e^x = \infty \iff x \text{ is neagtive} \end{cases}$$

For complex number γ

$$\begin{cases} e^\gamma \text{ converge} \iff Re\left(\gamma\right) < 0 \iff \gamma \text{ lies in OLHP} \\ e^\gamma \text{ diverge} \iff Re\left(\gamma\right) > 0 \iff \gamma \text{ lies in ORHP} \end{cases}$$

OLHP : Open left half plane , ORHP : Open right half plane. The plane is complex $z-$plane

Boundedness A signal / function $f(t)$ is bounded if

$$\exists M_f < \infty \; s.t. \; \left\|f(t)\right\| = \sup_{t \geq 0} \left|f(t)\right| < M_f$$

Absolutely Integrable A signal / function $g(t)$ is absolutely integrable if $\exists M < \infty$ s.t.

$$\int_0^\infty \left|g(t)\right| dt < M$$

Convergence with boundedness and monotonic For a function f

1. If it is bounded above / below
i.e. It has a lower bound or upper bound

2. It is monotonic increasing / decreasing
i.e. $\dfrac{df}{dt} > 0$ or $\dfrac{df}{dt} < 0$

Then it converge to the limit (the upper / lower bound)

1.2 Transfer Function

Linear ODEs from physical laws :

$$\sum_{k=0}^{m} a_k \frac{d^k x_I(t)}{dt^k} = \sum_{k=0}^{n} b_k \frac{d^k x_O(t)}{dt^k}$$

Apply Laplace Transfrom : $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots$, with assumption that all inital conditions are zero

$$\sum_{k=0}^{m} a_k s^k X_I(s) = \sum_{k=0}^{n} b_k s^k X_O(s)$$

The Transfer Function is thus

$$G(s) = \frac{X_O(s)}{X_I(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{n} b_k s^k}$$

Where the characteristic equation is

$$\sum_{k=0}^{n} b_k s^k = 0$$

2 Stability

- A system is *stable* if *every bounded* input produce a *bounded* output (BIBO Stable)
- A system with impulse response $g(t)$ is BIBO stable $\iff$ $g(t)$ is absolutely integrable
- A system with impulserepsonse $g(t)$ is BIBO stable $\iff \lim_{t\to\infty} g(t) = 0$
- For BIBO stable system, roots of the *characteristic equation* of the T.F. $G(s)$, should be lies on OLHP , i.e. negative real parts
- For the roots of the characteristic equation that have zero real parts, the system is *marginally stable* , but actually it is not stable for some specific input.

For the second last statment, the system is stable if the roots of $\sum_{k=0}^{n} b_k s^k = 0$ all lies in OLHP i.e.

$$\sum_{k=0}^{n} b_k s^k = b_k s^k + b_{k-1} s^{k-1} + \ldots + b_1 s + b_0 = 0$$

By Fundamental Theorem of algebra, this equation have exactly n roots,so it can be factorized into

$$\prod_{k=0}^{n} (s - p_k) = (s - p_1)(s - p_2)\ldots(s - p_n) = 0$$

i.e. The system is stable if $Re(p_k) < 0 \; \forall k$

This is VERY difficult. Since it require to solve a $n-$order polynomial. By some famous theorem, there is no $close\ form$ for the degree > 4.

Then it can only be solved numerically.

3 Routh - Hurwitz Stability Criterion

(Proof Skip)

For characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0$$

The *Routh Array*

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ s^0 & q & 0 & 0 & 0 \end{array} \quad \text{where} \begin{cases} b_1 = \dfrac{-\det\begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}}{a_{n-1}} & b_2 = \dfrac{-\det\begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}}{a_{n-1}} \\ c_1 = \dfrac{-\det\begin{pmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{pmatrix}}{b_1} & c_2 = \dfrac{-\det\begin{pmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{pmatrix}}{b_1} \end{cases} \cdots$$

Do the upper rowfirst , then the lower row

The system is stable ($\iff$ all $Re\,(p_k) < 0 \iff$ all poles lie on OLHP) iff the element of the first column of the Routh Array have same sign.

The number of sign change is the number of poles that lies on the ORHP

Special Case

1. The first element if zero

Replace the first element with $\epsilon > 0$

2. Whole row is zero

Replace the element by the coefficient of the auxiliary equation formed by the previous row.

Example 1. $G(s) = \dfrac{1}{as^2 + bs + c}$

Characteristic Equation : $as^2 + bs + c$

$$\begin{array}{c|ccc} s^2 & a & c & 0 \\ s^1 & b & 0 & 0 \\ s^0 & \dfrac{\det\begin{pmatrix} a & c \\ b & 0 \end{pmatrix}}{-b} = c & \dfrac{\det\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}}{-b} = 0 & \end{array}$$

If stable $\iff$ a, b, c same sign

Example 2. $G(s) = \dfrac{1}{as^3 + bs^2 + cs + d}$

Characteristic Eqaution : $as^3 + bs^2 + cs + d$

$$\begin{array}{c|ccc} s^3 & a & c & 0 \\ s^2 & b & d & 0 \\ s^1 & \dfrac{\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}}{-b} = \dfrac{bc - ad}{b} & 0 & \\ s^0 & \dfrac{\det\begin{pmatrix} b & d \\ \dfrac{bc-ad}{b} & 0 \end{pmatrix}}{-\left(\dfrac{bc-ad}{b}\right)} = d & c_2 & \end{array}$$

Stable if $a, b, d, \dfrac{bc - ad}{b}$ same sign

Example 3. $s^4 + s^3 - s - 1 = 0$

$$\begin{array}{c|ccc}
s^4 & 1 & 0 & -1 \\
s^3 & 1 & -1 & 0 \\
s^2 & \frac{\det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}}{-1} = 1 & \frac{\det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}{-1} = -1 & \\
s^1 & \frac{\det\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}{-1} = 0, \text{ replace by } \epsilon & \frac{\det\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}{-1} = 0 & \\
s^0 & \frac{\det\begin{pmatrix} 1 & -1 \\ \epsilon & 0 \end{pmatrix}}{-\epsilon} = \frac{-\epsilon(-1)}{-\epsilon} = -1 & &
\end{array}$$

The first column has 1 sign change, so the system is not stable

Example 4. $2s^3 + 4s^2 + 4s + 12 = 0$ $\qquad$ $s^3 + 2s^2 + 2s + 6 = 0$

$$\begin{array}{c|ccc}
s^3 & 2 & 4 & 0 \\
s^2 & 4 & 12 & 0 \\
s^1 & \frac{\det\begin{pmatrix} 2 & 4 \\ 4 & 12 \end{pmatrix}}{-4} = -2 & \frac{\det\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}}{-4} = 0 & \\
s^0 & \frac{\det\begin{pmatrix} 4 & 12 \\ -2 & 0 \end{pmatrix}}{2} = 12 & &
\end{array}
\qquad
\begin{array}{c|ccc}
s^3 & 1 & 2 & 0 \\
s^2 & 2 & 6 & 0 \\
s^1 & \frac{\det\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}}{-2} = -1 & \frac{\det\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}}{-2} = 0 & \\
s^0 & \frac{\det\begin{pmatrix} 2 & 6 \\ -1 & 0 \end{pmatrix}}{1} = 6 & &
\end{array}$$

They are all equivalent

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