Root Locus

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October 30, 2012

1 Review of related mathematics

1.1 Complex Algebra in Polar Form

For a complex number z , it can be expressed in polar form as

$$z = re^{j\theta}$$

Where $r = |z|$, $\theta = \tan^{-1}\dfrac{Im\,z}{Re\,z}$. Denote $r = |z|$ and $\theta = \angle z$

For multiple complex number

$$z_1 \times z_2 = |z_1|\angle z_1 \times |z_2|\angle z_2 = |z_1||z_2|\left(\angle z_1 + \angle z_2\right)$$

$$z_1 \div z_2 = |z_1|\angle z_1 \div |z_2|\angle z_2 = \frac{|z_1|}{|z_2|}\left(\angle z_1 - \angle z_2\right)$$

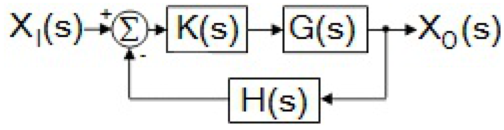
Similarly, for a function

$$z = \frac{\prod z_k}{\prod w_k} = \frac{\left(\prod |z_k|\right)\left(\sum \angle z_k\right)}{\left(\prod |w_k|\right)\left(\sum \angle w_k\right)} = \left(\frac{\prod |z_k|}{\prod |w_k|}\right)\left(\sum \angle z_k - \sum \angle w_k\right)$$

1.2 Roots of a equation

- Fundamental Theorem of Algebra : a order n polynomial equation has exactly n roots, including complex roots, equal roots, real roots.
- Complex roots occur in pairs
- If α is a double root of $f(x) = 0$, then it is also a root of $\dfrac{df(x)}{dx} = 0$

1.3 Review of transfer function



Consider a control system with plant $G(s)$, controller $K(s)$ and feedback $H(s)$, input $X_I(s)$ and output $X_O(s)$

- The open loop gain is $K(s)G(s)$
- The close loop gain is $\dfrac{K(s)G(s)}{1+K(s)G(s)H(s)}$
- Then the characteristic equation of the close loop gain is $1+K(s)G(s)H(s)$
- For simplicity , assume $K(s)$ is a number K
- The term$G(s)H(s)$ can be represented by a rational function $\dfrac{N(s)}{D(s)}$, and let the degree of $D(s)$ be n , and degree of $N(s)$ be m
- Let $G(s)H(s)$ has overall degree of p, so $p=n-m$. For proper fraction, $n \geq m$, i.e. $p \geq 0$
- By Fundamental Theorem of Algebra, and apply factorization, $G(s)H(s)=\dfrac{\sum_{k=1}^{m} b_k s^k}{\sum_{k=1}^{k=n} a_k s^k}=\dfrac{b_0\prod_{k=1}^{m}(s-z_k)}{a_0\prod_{k=1}^{n}(s-p_k)}$
- So the characteristic equation is $1+KG(s)H(s)=1+K\dfrac{b_0\prod_{k=1}^{m}(s-z_k)}{a_0\prod_{k=1}^{n}(s-p_k)}$
- Assume $K, b_0, a_0 > 0$

2 The Root Locus

The close loop transfer function $\dfrac{K(s)G(s)H(s)}{1+K(s)G(s)H(s)}$ contains the characteristic equation $1+K(s)G(s)H(s)$
For $K(s)=K$

$$1+KG(s)H(s)=1+K\frac{b_0\prod_{k=1}^{m}(s-z_k)}{a_0\prod_{k=1}^{n}(s-p_k)}=1+K\frac{b_0\,(s-z_1)\,(s-z_2)\ldots(s-z_m)}{a_0\,(s-p_1)\,(s-p_2)\ldots(s-p_n)}=0$$

i.e.

$$K\frac{b_0\,(s-z_1)\,(s-z_2)\ldots(s-z_m)}{a_0\,(s-p_1)\,(s-p_2)\ldots(s-p_n)}=-1$$

It can be seen that, as K varies, poles will change, the locus of poles when K changes is the root locus.

As this is a complex variable function, so we can consider the equality of both magnitude and phase of this equation

$$KG(s)H(s)=K\frac{N(s)}{D(s)}=-1 \quad \Rightarrow \begin{cases} \left|K\dfrac{N(s)}{D(s)}\right|=|-1| \\ \angle K\dfrac{N(s)}{D(s)}=\angle-1 \end{cases}$$

$$\because \begin{cases} K\in\mathbb{R}^+ \quad \Rightarrow \angle K=0^o \\ \angle-1=180^o+l\times 360^o \end{cases} \quad \Rightarrow \quad \begin{cases} K\left|\dfrac{N(s)}{D(s)}\right|=1 \\ \angle\dfrac{N(s)}{D(s)}=180^o+l\times 360^o \end{cases}$$

2.1 The Rules

2.1.1 Root locus is symmetryic about real axis

The characteristic equation, $D(s) = 0$ has real coefficients, so it's roots must occur in complex conjugate paris that symmetric about real axis. And the root locus is a diagram of root of characteristic equation (or equivalently poles of $G(s)$), so it is also symmetric about real axis.

2.1.2 Number of branches of root locus = order of characteristic equation (number of poles of $G(s)$)

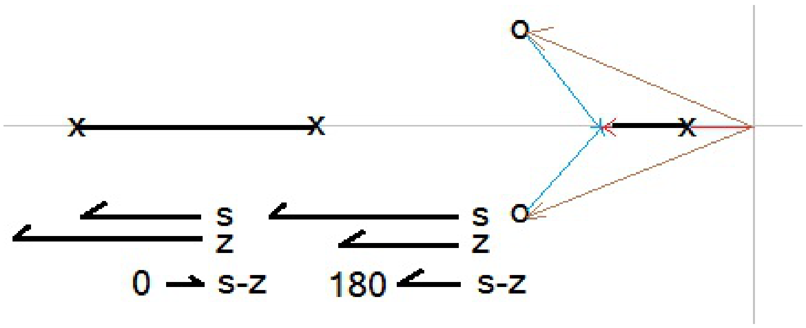
The root locus is locus of poles as K varies, so number of branches is number of poles (order of characteristic equation) , which is n

2.1.3 Root locus start at poles of G(s), and end at zeros of G(s).

$K\left|\frac{N(s)}{D(s)}\right| = 1 \quad \Rightarrow K\left|N(s)\right| = \left|D(s)\right|$, at the beginning, when $K = 0$, then $D(s) = 0$, i.e. the root locus start at poles of the characteristic equation.

$K\left|\frac{N(s)}{D(s)}\right| = 1 \quad \Rightarrow \left|\frac{N(s)}{D(s)}\right| = \frac{1}{K}$, at the end, when $K \to \infty$, then $N(s) = 0$, i.e. the root locus ends at zeros of the characteristic equation.

2.1.4 Segments on real axis that odd number of poles and zeros on its left are part of root locus



Start with $\angle\frac{N(s)}{D(s)} = 180^o + l \times 360^o$

$$\Longleftrightarrow \angle N(s) - \angle D(s) = 180^o + l \times 360^o = k \times 180^o \quad k \text{ is odd}$$

$$\Longleftrightarrow \sum\angle(s - z_k) - \sum\angle(s - p_k) = k \times 180^o$$

From the diagram, a z that is on the left of the s , its resultant $s - z$ has angle 0^o
From the diagram, a z that is on the right of the s , its resultant $s - z$ has angle 180^o
Using same logic to poles, then
The term $\sum\angle(s - z_k) - \sum\angle(s - p_k)$ means the angle differnece between all the zero and poles, since any zero or pole on the left of the s result in 0^o , so only consider the zero and pole of the right side of s
Since one zero on the right side of s contribute to 180^o and same as the pole does. But since the angle $(s - p)$ is negative, so it means subtract 180^o .

By the $\sum\angle(s - z_k) - \sum\angle(s - p_k) = k \times 180^o$, k is odd number, then

Segments on real axis that odd number of poles and zeros on its right are part of root locus

For poles and zeros not on the real axis, from the diagram, the angle cancel out by the conjugate pairs, so those are not counted.

2.1.5 The asymptotes as $s \to \infty$ is in $\alpha = \dfrac{\sum p_i - \sum z_i}{q}$, $\phi = \dfrac{180^o + l \cdot 360^o}{q}$

Consider the characterisitic eqaution

$$1 + KG(s)H(s) = 1 + K\frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n}$$

Take $\dfrac{s^m}{s^n}$ out

$$1+KG(s)H(s) = 1+K\frac{s^m}{s^n}\frac{b_0 + \frac{b_1}{s} + ... + \frac{b_m}{s^m}}{a_0 + \frac{a_1}{s} + ... + \frac{a_n}{s^n}} = 1+K\frac{1}{s^{n-m}}\frac{b_0 + \frac{b_1}{s} + ... + \frac{b_m}{s^m}}{a_0 + \frac{a_1}{s} + ... + \frac{a_n}{s^n}} = 1+\frac{K}{s^q}\frac{b_0 + \frac{b_1}{s} + ... + \frac{b_m}{s^m}}{a_0 + \frac{a_1}{s} + ... + \frac{a_n}{s^n}}$$

Thus

$$\lim_{s\to\infty} 1 + KG(s)H(s) = 1 + \frac{K}{s^q}\frac{b_0}{a_0} = 1 + K\frac{b_0}{a_0}s^{-q}$$

Then consider the phase

$$\angle\left(\lim_{s\to\infty} 1 + KG(s)H(s)\right) = \angle\left(\lim_{s\to\infty} 1 + KG(s)H(s)\right)$$

$$= \angle\left(\lim_{s\to\infty} 1 + K\frac{N(s)}{D(s)}\right)$$

$$= \angle\left(1 + K\frac{b_0}{a_0}s^{-q}\right)$$

$$= \angle\left(1 + \underbrace{K\frac{b_0}{a_0}|s|^{-q}}_{\mathbb{R}^+} e^{jq\phi}\right)$$

By $\angle\mathbb{R}^+$ can be ignore and $\angle\dfrac{N(s)}{D(s)} = 180^o + l \times 360^o and$

$$\angle\left(e^{jq\phi}\right) = q\phi = 180^o + l \times 360^o$$

$$\therefore \phi = \frac{180^o + l \times 360^o}{q}$$

2.1.6 The breakaway / break in points

When more than one branches meet in the break-away / break-in point, that point is a multiple root of $1 + KG(s)H(s) = 0$

As the break-away / break-in point is a multiple root of $1 + KG(s)H(s) = 0$, so it is also a root of $\frac{d}{ds}KG(s)H(s) = 0$

Let the break away / break in point be x

$$1 + KG(x)H(x) = 0 \quad \Rightarrow 1 + K\frac{N(x)}{D(x)} = 0 \quad \Rightarrow \begin{cases} D(x) + KN(x) = 0 \\ D'(x) + KN'(x) = 0 \end{cases}$$

Eliminate the K

$$K = -\frac{D(x)}{N(x)} = -\frac{D'(x)}{N'(x)} \qquad \Rightarrow D(x)N'(x) = D'(x)N(x) \quad \Rightarrow N'(x)D(x) - D'(x)N(x) = 0$$

Consider

$$\frac{d}{ds}G(s)H(s) = \frac{d}{ds}\frac{N(s)}{D(s)} = \frac{N'(s)D(s) - N'(s)D(s)}{D^2(s)}$$

Plug in the equation $N'(x)D(x) - D'(x)N(x) = 0$, so to find the break-in / break-away point is to solve

$$\frac{d}{ds}G(s)H(s) = 0$$

2.1.7 If the root locus corsses the stability boundary (the imaginary axis), the corssing point can be found by solving $1 + KG(j\omega)H(j\omega) = 0$

Recall that , the location of the poles of the system T.F. determine the damping property of the system.
For stable system, the poles is in LHP (Left Half Plane)

When the poles lies on the imaginary axis, (i.e. real part is zero) , the system output is no-damped oscillation, it is the boundary case for stable system.

When the poles has postive real part $e^{pt} = e^{Re(p)t}e^{jIm(p)t} \to \infty$ as $t \uparrow$, so the system is unstable.
So the boundary stable case can be found by solving

$$1 + KG(s)H(s) = 0 \quad s \text{ lies on imaginary axis}$$

i.e.

$$1 + KG(j\omega)H(j\omega) = 0$$

$-END-$