Cauchy Index of Tangent of the Phase Angle

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1 Review of Related Mathematics

1.1 Complex Algebra - Polar Form

$\forall z : z = x + jy \in \mathbb{C}$, this rectangular form can be represented by polar form :

$z = \rho e^{j\theta}$ where $\rho = \sqrt{x^2 + y^2} = \sqrt{[Re(z)]^2 + [Im(z)]^2}$ and $\theta = \tan^{-1} \dfrac{y}{x} = \tan^{-1} \dfrac{Im(z)}{Re(z)}$

For $z = \prod_{k=1}^{n} z_k = z_1 z_2 ... z_n$, their polar form is $z = \rho \prod_{k=1}^{n} \exp(j\theta_k) = \rho e^{j\theta_1} e^{j\theta_2} ... e^{j\theta_n} = \rho e^{j(\theta_1 + \theta_2 + ... + \theta_n)}$

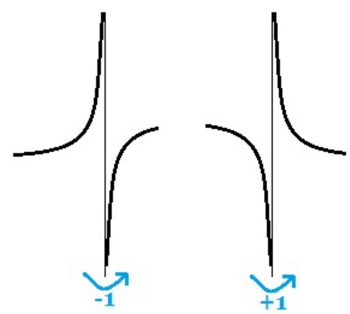
i.e.

$$\theta = \sum_{k=1}^{n} \theta_k$$

1.2 Cauchy Index

$$\text{Cauchy Index} = \begin{cases} +1 & -\infty \to +\infty \\ -1 & +\infty \to -\infty \\ 0 & \text{else} \end{cases}$$

$$\iff \begin{cases} +1 & \lim_{x \to a^-} f(x) = -\infty \cap \lim_{x \to a^+} f(x) = +\infty \\ -1 & \lim_{x \to a^-} f(x) = +\infty \cap \lim_{x \to a^+} f(x) = -\infty \\ 0 & \text{else} \end{cases}$$



2 The Cauchy Index of Tangent of Phase Angle

For a function $f(z) = Re[f(z)] + Im[f(z)]$

$$f(z) = \sum_{k=1}^{n} a_k z^{n-k} = a_0 z^n + a_1 z^{n-1} + ... + a_n$$

By Fundamental Theorem of Algebra

Degree n equation has exactly n roots

$\therefore$

$$f(x) = \prod_{k=1}^{n} (z - r_k) = (z - r_k)(z - r_1)(z - r_2) ... (z - r_n)$$

Assume no roots of $f(z) = 0$ lie on imaginary axis

Let

$$N = \text{number of roots of } f(z) = 0 \text{ with negative real parts}$$
$$P = \text{number of roots of } f(z) = 0 \text{ with positive real parts}$$

And thus

$$N + P = n$$

As $f(z)$ is complex, express $f(z)$ in polar form

$$f(z) = \rho(z)e^{j\theta(z)}$$

Where

$$\rho(z) = \sqrt{Re^2\left[f(z)\right] + Im^2\left[f(z)\right]} \qquad\qquad \theta\left(z\right) = \tan^{-1}\frac{Im\left[f(z)\right]}{Re\left[f(z)\right]}$$

Since $f(z) = \prod_{k=1}^{n}\left(z - r_k\right) = \left(z - r_1\right)\left(z - r_2\right)...\left(z - r_n\right)$, therefore

$$\theta(z) = \sum_{k=1}^{n}\theta_{r_k}(z) = \theta_{r_1}(z) + \theta_{r_2}(z) + ... + \theta_{r_n}(z)$$

Where

$$\theta_{r_i}(z) = \angle\left(z - r_i\right) = \tan^{-1}\frac{Im\left(z - r_i\right)}{Re\left(z - r_i\right)}$$

Case 1 . For the i^{th} root of $f(z)$, it it has a positive real part, i.e. $Re\left(r_i\right) > 0$ then

$$\begin{cases} \theta_{r_i}(z)\Big|_{z=j\infty} = \angle\left(z - r_i\right)\Big|_{z=0+j\infty} = \tan^{-1}\dfrac{\infty - Im\left(r_i\right)}{0 - Re\left(r_i\right)} = \tan^{-1}\dfrac{\infty}{-Re\left(r_i\right)} = -\dfrac{\pi}{2} \\ \theta_{r_i}(z)\Big|_{z=-j\infty} = \angle\left(z - r_i\right)\Big|_{z=0-j\infty} = \tan^{-1}\dfrac{-\infty - Im\left(r_i\right)}{0 - Re\left(r_i\right)} = \tan^{-1}\dfrac{\infty}{Re\left(r_i\right)} = +\dfrac{\pi}{2} \end{cases}$$

Case 2. For the i^{th} root of $f(z)$, it it has a negative real part , i.e. $Re\left(r_i\right) < 0$ then

$$\begin{cases} \theta_{r_i}\left(z\right)\Big|_{z=j\infty} = \angle\left(z - r_i\right)\Big|_{z=0+j\infty} = \tan^{-1}\dfrac{\infty - Im\left(r_i\right)}{0 - Re\left(r_i\right)} = \tan^{-1}\dfrac{\infty}{\left|Re\left(r_i\right)\right|} = +\dfrac{\pi}{2} \\ \theta_{r_i}\left(z\right)\Big|_{z=-j\infty} = \angle\left(z - r_i\right)\Big|_{z=0-j\infty} = \tan^{-1}\dfrac{-\infty - Im\left(r_i\right)}{0 - Re\left(r_i\right)} = \tan^{-1}\dfrac{-\infty}{\left|Re\left(r_i\right)\right|} = -\dfrac{\pi}{2} \end{cases}$$

Therefore,

$$\theta(z)\Big|_{z=j\infty} = \theta_{r_1}(z)\Big|_{z=j\infty} + \theta_{r_2}(z)\Big|_{z=j\infty} + ... + \theta_{r_n}(z)\Big|_{z=j\infty} = N\frac{\pi}{2} + P\left(-\frac{\pi}{2}\right)$$

$$\theta(z)\Big|_{z=-j\infty} = \theta_{r_1}(z)\Big|_{z=-j\infty} + \theta_{r_2}(z)\Big|_{z=-j\infty} + ... + \theta_{r_n}(z)\Big|_{z=-j\infty} = N\left(-\frac{\pi}{2}\right) + P\left(\frac{\pi}{2}\right)$$

Recall that, N , P are the number of roots that has negative real part and positive real parts, and $N + P = n$.

Now let

$$\Delta = \frac{1}{\pi}\theta(z)\Big|_{-j\infty}^{j\infty}$$

Then

$$\Delta = \frac{1}{\pi}\theta(z)\Big|_{-j\infty}^{j\infty} = \frac{1}{\pi}\left\{\left[N\frac{\pi}{2} - P\frac{\pi}{2}\right] - \left[-N\frac{\pi}{2} + P\frac{\pi}{2}\right]\right\} = N - P$$

With $N + P = n$

$$2N = n + \Delta \qquad \qquad N = \frac{n + \Delta}{2}$$

$$or$$

$$2P = n - \Delta \qquad \qquad P = \frac{n - \Delta}{2}$$

Thus, given a polynomial function $f(z)$ with degree n , N , P can be found by evalute Δ

And the value Δ is the Cauchy Index for the function $\theta(z)$

i.e. , the value for the Tangent of the Phase Angle of a function $f(z)$

$-END-$