Asymptotic Bode Plot & Lead-Lag Compensator

Ang Man Shun

2012-12-15

1. Introduction

Consider a general transfer function

$$G(s) = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{m} b_k s^k} = A \frac{\prod_{k=0}^{n}(s - z_k)}{\prod_{k=0}^{m}(s - p_k)} \quad m > n$$

When $s = j\omega$, transfer function $G(j\omega)$ is very useful in frequency response analysis. As now the transfer function, or using another name, the *gain* is a function of frequency ω , it means that "amplification ability " of such system change as ω change, where the ω is the input or operating frequency.

There are some methods to investigate the behaviour of such "amplification ability changes" : Bode Plot, Nyquist Plot, Nichol Plot.

The *Bode Plot* consists of 2 plots : the magnitude plot and the phase plot, it is an *asymptotic* plot, an approximation plot.

The magnitude Bode Plot is 20log|G(jω)| (y-axis) vs ω(x-axis, log scale)

The magnitude axis is expressed in dB of powers, it is a convention.

The frequency axis is expressed in log scale, it is also a convention, with a very good reason.

2. Review of some related mathematics

Log $\log \dfrac{AB}{CD} = \log A + \log B - \log C - \log D \qquad \log \left|\dfrac{AB}{CD}\right| = \log|A| + \log|B| - \log|C| - \log|D|$

Standard Form $\log|j\omega + a| = \log\left|a\left(\dfrac{j\omega}{a} + 1\right)\right| = \log|a| + \log\left|\dfrac{j\omega}{a} + 1\right|$

Approximation Consider $\log\left|\dfrac{j\omega}{a} + 1\right|$, a is called the *corner frequency*

$$\text{when} \begin{cases} \omega \ll a \\ \omega \gg a \end{cases} \Rightarrow \begin{cases} \dfrac{\omega}{a} \ll 1 \\ \dfrac{\omega}{a} \gg 1 \end{cases} \Rightarrow \begin{cases} \left|\dfrac{j\omega}{a} + 1\right| \approx |1| = 1 \\ \left|\dfrac{j\omega}{a} + 1\right| \approx \left|\dfrac{j\omega}{a}\right| = \dfrac{\omega}{a} \end{cases} \Rightarrow \begin{cases} \log\left|\dfrac{j\omega}{a} + 1\right| \approx 0 \\ \log\left|\dfrac{j\omega}{a} + 1\right| \approx \log\dfrac{\omega}{a} \end{cases}$$

$$\Longleftrightarrow \begin{cases} 20\log\left|\dfrac{j\omega}{a} + 1\right| = 0 & \text{Flat horizontal line} \\ +20\log\left|\dfrac{j\omega}{a} + 1\right| = +20\log\dfrac{\omega}{a} & \text{Stright line increasing with 20dB per decade} \end{cases}$$

Arc-Tan

$$\tan^{-1}\infty = \pm\frac{\pi}{2} \qquad \tan^{-1} 0 = 0$$

3. Example

For example, consider a degree 1 transfer function with 5 zeros and poles.

$$G(s) = \frac{A\,(s+a)\,(s+b)}{(s+c)\,(s+d)\,(s+e)}$$

Where $-a$, $-b$ are zeros, $-c$, $-d$, $-e$ are poles.

Before doing the Bode Plot analysis, turn the transfer function into "Standard Form"

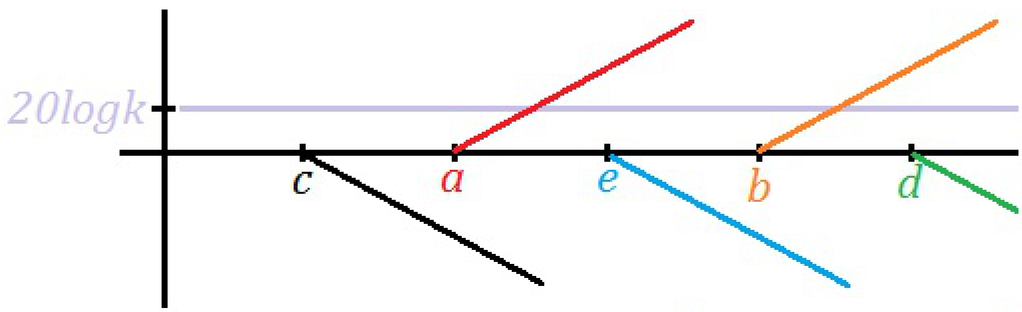
$$G(s) = \frac{Aab}{cde} \cdot \frac{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)\left(\frac{s}{e}+1\right)} = \frac{k\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)\left(\frac{s}{e}+1\right)} \qquad k = \frac{Aab}{cde}$$

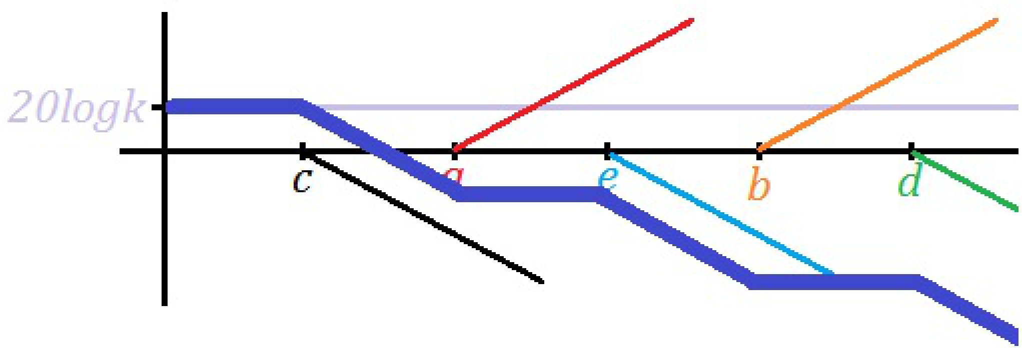
The magnitude (in dB of powers) is thus

$$20\log|G(s)| = 20\log\left|\frac{k\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)\left(\frac{s}{e}+1\right)}\right|$$

$$= 20\log|k| + 20\log\left|\frac{s}{a}+1\right| + 20\log\left|\frac{s}{b}+1\right| - 20\log\left|\frac{s}{c}+1\right| - 20\log\left|\frac{s}{d}+1\right| - 20\log\left|\frac{s}{e}+1\right|$$

The Bode Plot is thus





4. For more general case

- The Effect of DC term : a horizontal line
- The Effect of a pole at α : For $\omega < \alpha$, no effect (since it is in log scale !) , for $\omega > \alpha$, it goes down at a rate of 20dB per decade.
- The Effect of a zero at β : For $\omega < \beta$, no effect (since it is in log scale !) , for $\omega > \beta$, it goes up at a rate of 20dB per decade.

More General

- For a *simple* , 1^{st} order pole at α : For $\omega < \alpha$, no effect, for $\omega > \alpha$, it goes down at a rate of $1 \cdot 20$dB per decade.
- For a n^{th} order pole at β : For $\omega < \alpha$, no effect, for $\omega > \alpha$, it goes down at a rate of $n20$ dB per decade.
- In same logic, a m^{th} order zero at γ will give a increasing straight with slope of $m20$dB per decade , starting at $\omega = \gamma$

5. Summary of Magnitude Plot

The algorithm to draw Magnitude Bode Plot is

- Turn the transfer function into standard form $(s+a) \rightarrow a(\frac{s}{a}+1)$
- Find all the corner frequency
- For zeros, the lines go up. For poles, the lines go down.
- The slope of the line is $20dB \cdot$the degree of the pole/zero.

6. Phase Plot

Consider $(j\omega + a)$, the angle of this term is $\tan^{-1}\frac{\omega}{a}$

When $\omega \ll a$ (The axis is in scale of decade, i.e. 10 time)

$$\omega = 0.1a \qquad \tan^{-1}\frac{\omega}{a} = \tan^{-1}(0.1) = 5.7^o$$

$$\omega = 0.01a \qquad \tan^{-1}\frac{\omega}{a} = \tan^{-1}(0.01) = 0.57^o$$

When $\omega \gg a$ (The axis is a log scale, so consider a decade , i.e. 10 time)

$$\omega = 10a \qquad \tan^{-1}\frac{\omega}{a} = \tan^{-1}(10) = 84.3^o$$

$$\omega = 100a \qquad \tan^{-1}\frac{\omega}{a} = \tan^{-1}(100) = 89.4^o$$

So in general,

$$\tan^{-1}\frac{\omega}{a} \approx \begin{cases} 0^o & \omega \ll a \\ \\ 90^o & \omega \gg a \end{cases}$$

7. Examples

$$G(s) = \frac{K_0}{T_0 s + 1} \qquad K_0, T_0 > 0$$

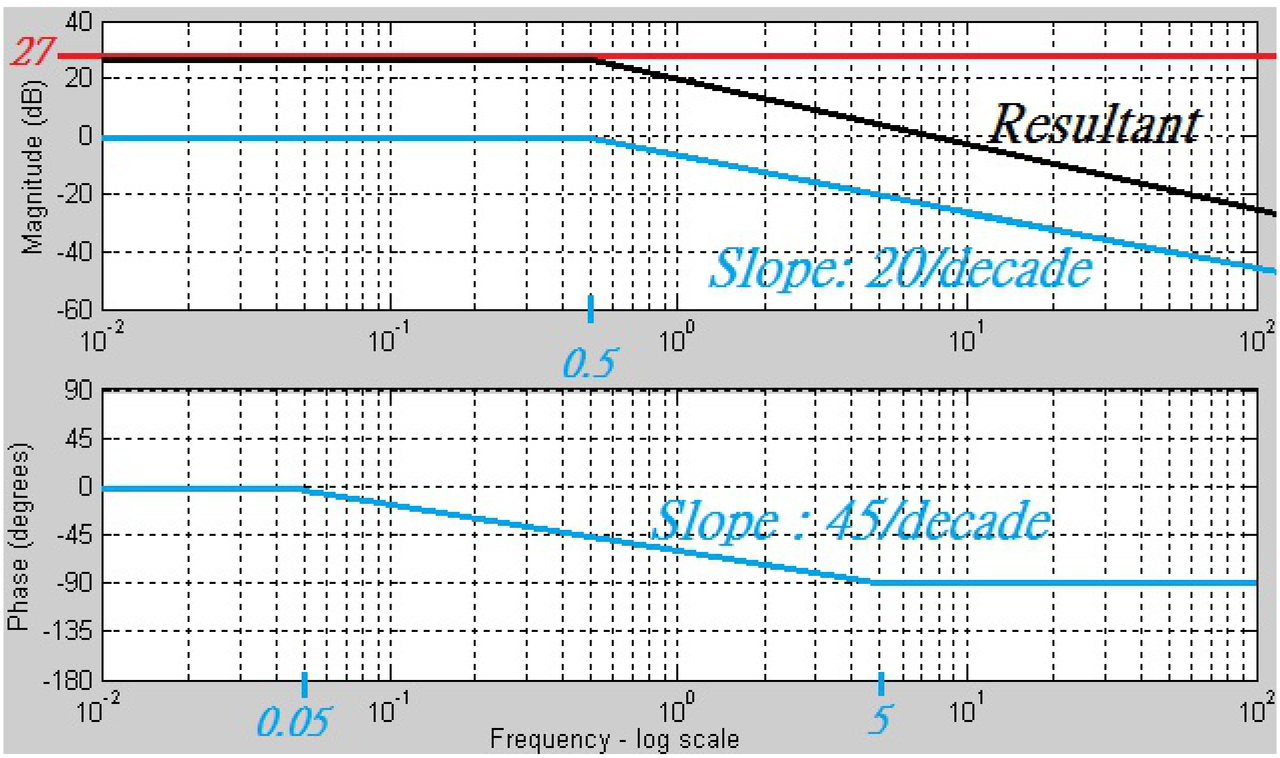
$$20\log|G(j\omega)| = 20\log|K_0| - 20\log\left|\frac{j\omega}{1/T_0} + 1\right|$$

$$\angle G(j\omega) = \underbrace{\angle K_0}_{0} - \angle\left(\frac{j\omega}{1/T_0} + 1\right)$$

Let $K = 4$, $T_0 = 0.5$

$$G(s) = \frac{4}{\frac{1}{2}s + 1}$$

- $20\log|4| = 27$
- Corner Frequenct in magnitude plot : 0.5
- Corner Frequency in phase plot : 0.05 and 5



8. Example 2

$$G(s) = \frac{K_0}{(s+a)\,(s+b)^2} \qquad K_0, a, b > 0$$

First change it into standard form

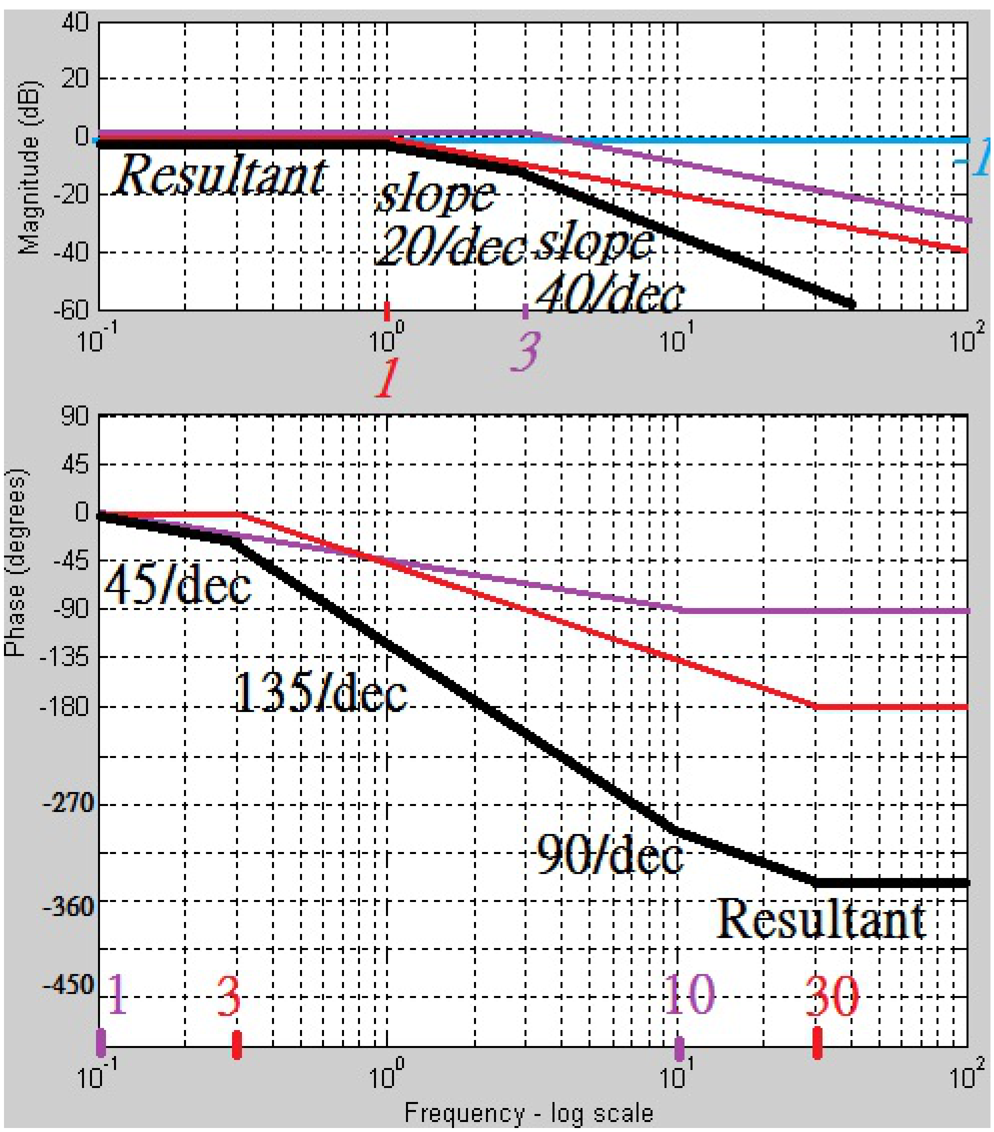
$$G(s) = \frac{\dfrac{K_0}{ab^2}}{\left(\dfrac{s}{a}+1\right)\left(\dfrac{s}{b}+1\right)^2}$$

$$20\log|G(j\omega)| = 20\log\left|\frac{K_0}{ab^2}\right| - 20\log\left|\frac{j\omega}{a}+1\right| - 2\times 20\log\left|\frac{j\omega}{b}+1\right|$$

$$\angle G(j\omega) = \underbrace{\angle\left(\frac{K_0}{ab^2}\right)}_{0} - \angle\left(\frac{j\omega}{a}+1\right) - 2\angle\left(\frac{j\omega}{b}+1\right)$$

Let $K_0 = 8$, $a = 1$, $b = 3$

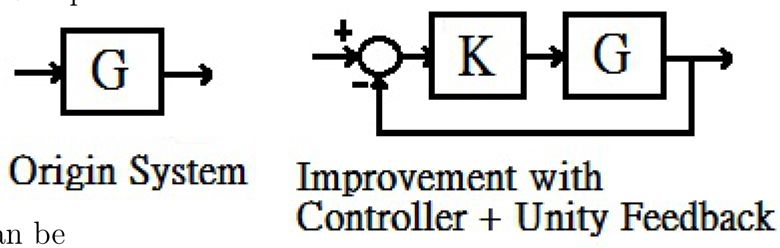
- $20\log\left|\frac{8}{9}\right| = -1$
- Corner Frequency in magnitude plot : 1 , 3
- Corner Frequency in phase plot : 0.1 & 10 , 0.3 & 30



9. Lead Lag Compensator $\dfrac{1+Ts}{1+\alpha Ts}$

Purpose of Compensator

System is not always perfect naturally, some improvements can be made by adding a special type of controller, the Lead-Lag Compensator



The improvements can be

- Increase Gain
- Reduce steady state Error
- Force he system response faster
- Force the system become less oscillatory
- Increase bandwidth
- Improve stability margin

The Lead and the Lag Compensator

There are two form, both are the same

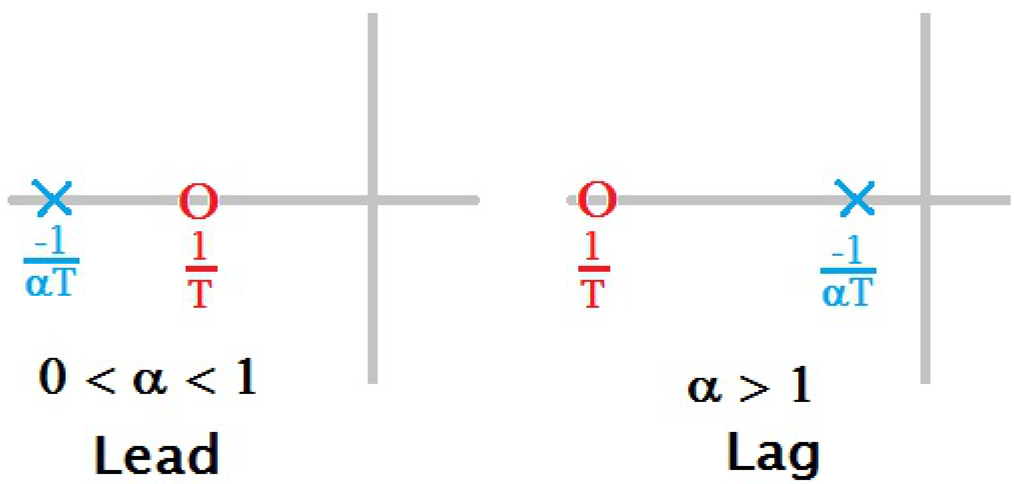
$$\frac{1+\alpha Ts}{1+Ts} \qquad or \qquad \frac{1+Ts}{1+\alpha Ts}$$

Consider the second one

$$K_{DC}\frac{1+Ts}{1+\alpha Ts}$$

K_{DC} : DC gain , when $G(s)_{s=0}$

The Transfer Function of the compensator has 1 zero and 1 poles : $\dfrac{-1}{T}$ and $\dfrac{-1}{\alpha T}$
The α is key parameter here.



Consider their Bode Plot

$$K(s) = K_{DC}\frac{1+Ts}{1+\alpha Ts}$$

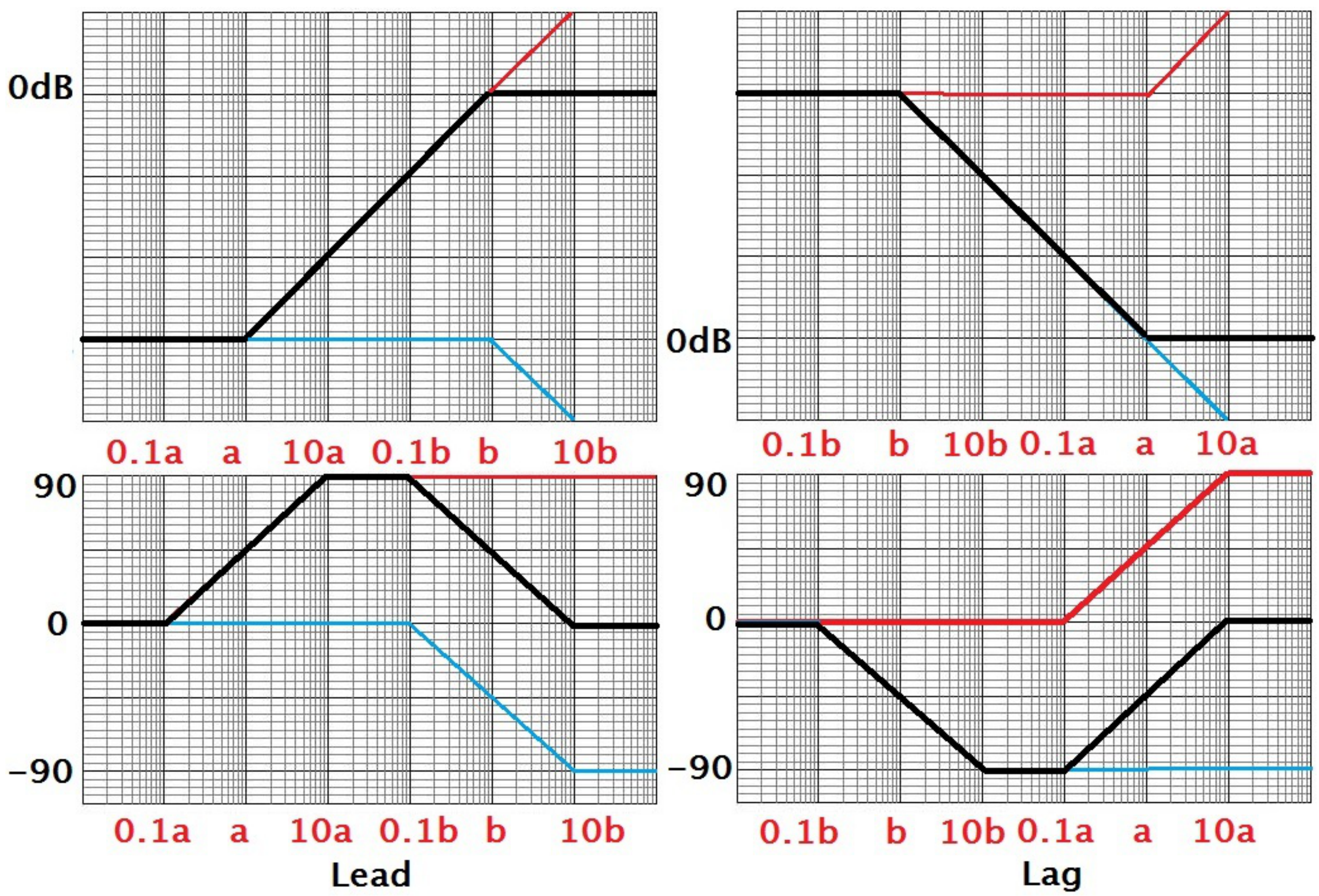
Standard Form

$$K(s) = K_{DC}\frac{1+\frac{s}{1/T}}{1+\frac{s}{1/\alpha T}}$$

$$20\log|K(j\omega)| = 20\log|K_{DC}| + 20\log\left|1+j\frac{\omega}{1/T}\right| - 20\log\left|1+j\frac{\omega}{1/\alpha T}\right|$$

$$\angle K(j\omega) = \underbrace{\angle K_{DC}}_{0} + \angle\left(1+j\frac{\omega}{1/T}\right) - \angle\left(1+j\frac{\omega}{1/\alpha T}\right)$$

- Corner frequency of magnitude plot : $\frac{1}{T}$, $\frac{1}{\alpha T}$
- Corner frequency of phase plot : $\frac{1}{10T}$ & $\frac{10}{T}$ and $\frac{1}{10\alpha T}$ & $\frac{10}{\alpha T}$
- Denote $\frac{1}{T} = a$ $\frac{1}{\alpha T} = b$
- The following Bode Plot ignore the DC part (Assume $K_{DC} = 1$)



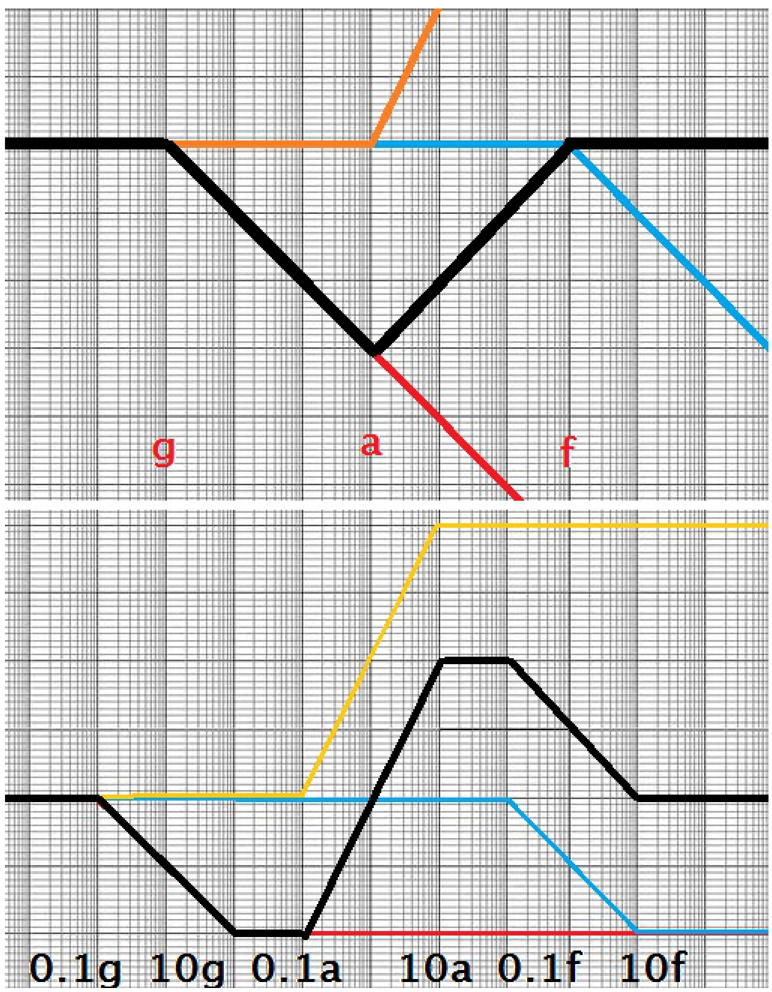
Lead-Lag compensator

$$K(s) = K_{DC} \cdot \frac{1+Ts}{1+\alpha_1 Ts} \cdot \frac{1+Ts}{1+\alpha_2 Ts}$$

$$20\log|K| = 20\log|K_{DC}| + 2\times 20\log\left|1+j\frac{\omega}{1/T}\right| - 20\log\left|1+j\frac{\omega}{1/\alpha_1 T}\right| - 20\log\left|1+j\frac{\omega}{1/\alpha_2 T}\right|$$

$$\angle K = 2\angle\left(1+j\frac{\omega}{1/T}\right) - \angle\left(1+j\frac{\omega}{1/\alpha_1 T}\right) - \angle\left(1+j\frac{\omega}{1/\alpha_2 T}\right)$$

- Corner frequency in magnitude plot : $\frac{1}{T}$ (denote as a) , $\frac{1}{\alpha_1 T}$ denote as f , $\frac{1}{\alpha_2 T}$ denote as g
- $g < a < f$
- Corner frequency in phase plot : $0.1a, 10a$, $0.1f, 10f$, $0.1g, 10g$



$-END-$