State Space Notation

Ang Man Shun

2012-11-14

State Space Analysis is a method that turning the original problems of equations of dynamic variables, usually expressed in differential equations (order $\geq$1), into a groups of $first\ order\ ODEs$, and then grouping them into one single equation, and finally handle it using ordinary control theory method via the techniques of linear algebra. This method is very useful in multiple-input multiple-output system analysis.

1 Review of related Mathematics notations

1. ODEs

$$\left(\frac{d^k y(t)}{dt^k}\right)^m = a_1x_1(t) + a_2x_2(t) + ... + a_hx_h(t)$$

Is a k-order , degree m ODE with h terms.

2. Matrix

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1b} \\ a_{21} & a_{22} & ... & a_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ a_{a1} & a_{a2} & ... & a_{ab} \end{pmatrix}_{ab}$$

Is a $a \times b$ matrix : a horizontal row vector or b vertical column vector.

3. System of linear equations

$$\begin{aligned} x = a\theta + b\phi + c\psi \\ y = m\theta + n\phi + p\psi \\ z = r\theta + s\phi + t\psi \end{aligned} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ m & n & p \\ r & s & t \end{bmatrix} \begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix}$$

2 Terms

- The variables in the differential equations are called *state* , they contain the information of the system.
- A system described by n^{th} order ODEs will have n *independent* state variable.
- The selection of state variable are arbitary, and *not unique*

- Sometimes that the variable are even only conceptual, mathematical, not physically realizable
- State variable are denoted as $x_1(t)$, $x_2(t)$, ... , $x_n(t)$
- The vector that compose of the n state variables are called *state vector* $\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad or \qquad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix}^T$$

- Thus the *space* constructed using the state variables x_1,x_2,...,x_n as coordinate axis are the *state space*
- As time propagate, $\mathbf{x}(t)$ will give a curve / locus on the state space, the *state locus*

3 The State Equation

3.1 Single-Input Single-Output System

Consider the State Space Modeling of a LTI n^{th}-ODE Single-Input Single-Output System

As this is a n^{th} order system, so there are n state vairables : $x_1(t)$, $x_2(t)$, ... , $x_n(t)$, with input x_I and output x_O , the general state eqaution as

$$\frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + b_1 x_I(t)$$

$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + b_2 x_I(t)$$

$$...$$

$$\frac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + b_n x_I(t)$$

The state equations in ODE Form

The system will also have a general output equation as

$$x_O(t) = c_1 x(t) + c_2 x(t) + ... + c_n x_n(t)$$

By using Vector-Matrix Notation, the equations can be expressed in a more compact form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} x_I \qquad x_O = \begin{bmatrix} c_1 & c_2 & ... & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The state equations and output equation in Matrix Form

Or in further compact (with subscript denoting the dimension)

$$\dot{\mathbf{x}}_n = \mathbf{A_{n\times n}x_n} + \mathbf{b_n x_I} \qquad \mathbf{x_O} = \mathbf{c_n x_n}$$

3.2 Multiple-Input Multiple-Output System

Consider the State Space Modeling of a LTI n^{th}-ODE r-Input m-Output System

- As this is a n^{th} order system, so there are n state vairables : $x_1(t)$, $x_2(t)$, ... , $x_n(t)$
- As there are r input, there are r input variables : $u_1(t)$, $u_2(t)$, ... , $u_r(t)$
- As there are m output, there are m output variables : $y_1(t)$, $y_2(t)$, ... , $y_m(t)$

Now the general state eqaution as

$$\frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) + b_1u_1(t) + b_2u_2(t) + ... + b_ru_r(t)$$

$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + b_1u_1(t) + b_2u_2(t) + ... + b_ru_r(t)$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + b_1u_1(t) + b_2u_2(t) + ... + b_ru_r(t)$$

The general $m-$output equation is

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + ... + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + ... + d_{1r}u_r(t)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + ... + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + ... + d_{2r}u_r(t)$$

$$\vdots$$

$$y_n(t) = c_{m1}x_1(t) + c_{m2}x_2(t) + ... + c_{mn}x_n(t) + d_{m1}u_1(t) + d_{m2}u_2(t) + ... + d_{mr}u_r(t)$$

Although using summation notation can make the equation more compact, but matrix notation is more intuitive.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & ... & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & ... & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & ... & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

The expanded form can be denoted as

$$\dot{\mathbf{x}}_{n\times 1} = \mathbf{A_{n\times n}x_{n\times 1}} + \mathbf{B_{n\times r}u_{r\times 1}} \qquad \mathbf{y_{m\times 1}} = \mathbf{C_{m\times n}x_{n\times 1}} + \mathbf{D_{m\times r}u_{r\times 1}}$$

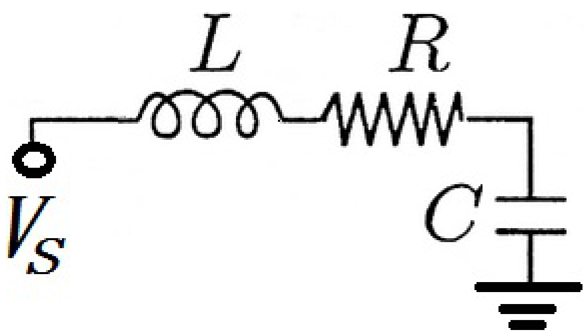
Or just

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are rectangular matrix with different sizes
- $\mathbf{x}, \mathbf{y}, \mathbf{u}$ are column vector with different size
- $\mathbf{x_{n\times 1}}$ is state vector , $\mathbf{u}_{r\times 1}$ is input vector, $\mathbf{y_{m\times 1}}$ is output vector
- $\mathbf{A_{n\times n}}$ is system coefficient matrix, $\mathbf{B_{n\times r}}$ is input control matrix, $\mathbf{C}_{m\times n}$ is output matrix, $\mathbf{D_{m\times r}}$ is direct feedthroug matrix

As the selection of state variable is not unique, thus the state equation is also not unique. But different form of state equations are related by using linear transforms

4 LRC Circuit Example



The variables : i , v_L , v_R , v_C, v_S , take v_C and i as the state variables, v_s as input , and v_c as output : $x_1 = i$, $x_2 = v_c$, $u = v_s$, $y = v_c$

The equations $v_s = v_L + v_R + v_C = L\dfrac{di}{dt} + iR + v_C$, $q_c = Cv_c \quad \Longleftrightarrow \quad i = C\dfrac{dv_C}{dt}$

Thus

$$\overset{\bullet}{x}_1 = \frac{di}{dt} = \frac{1}{L}v_s - \frac{R}{L}i - \frac{1}{L}v_C = \frac{1}{L}u - \frac{R}{L}x_1 - \frac{1}{L}x_2 \qquad \overset{\bullet}{x}_2 = \frac{dv_c}{dt} = \frac{1}{C}i = \frac{1}{C}x_1$$

Using matrix notation

$$\begin{bmatrix} \overset{\bullet}{x}_1 \\ \overset{\bullet}{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} u \qquad\qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$-END-$