ELEC2205/3245. Step-by-Step demonstration of convolution integral calculation

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Question:

3. A system has an impulse response given by

$$g(t) = 1(t)(1 - e^{-t})$$

Find the response $y(t)$ of the system to the input

$$u(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}$$

The equation of convolution is $y(t) = x(t) * h(t)$. By definition, it has the form as

$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau$$

The input signal $x(t)$ is $u(t)$. Therefore $x(\tau) = u(\tau)$. Impulse response is $h(t)$ is $g(t)$, therefore $h(t-\tau) = g(t-\tau)$, which is

$$g(t-\tau) = 1(t-\tau)(1 - e^{-(t-\tau)})$$

The $1(t-\tau)$ is the "step function". That is

$$1(t) = \begin{cases} 1 & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

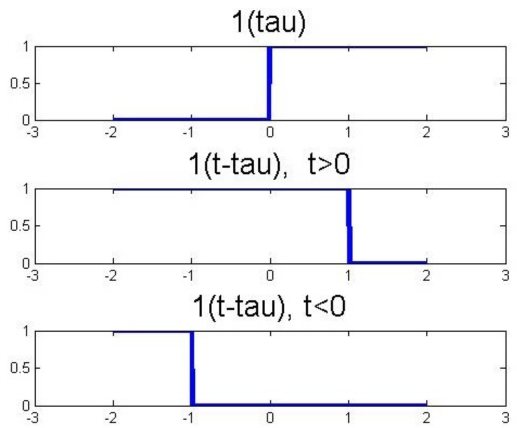
(Note. Some books use u(t) as step function.)

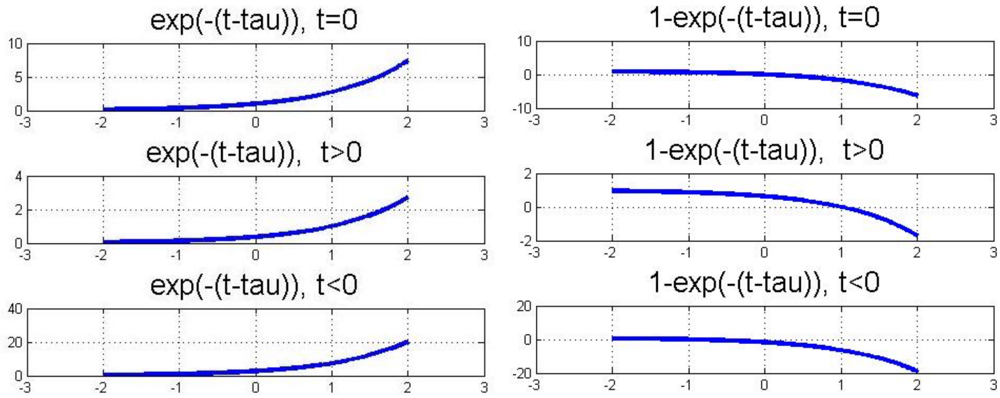
Thus the integral now has the following form

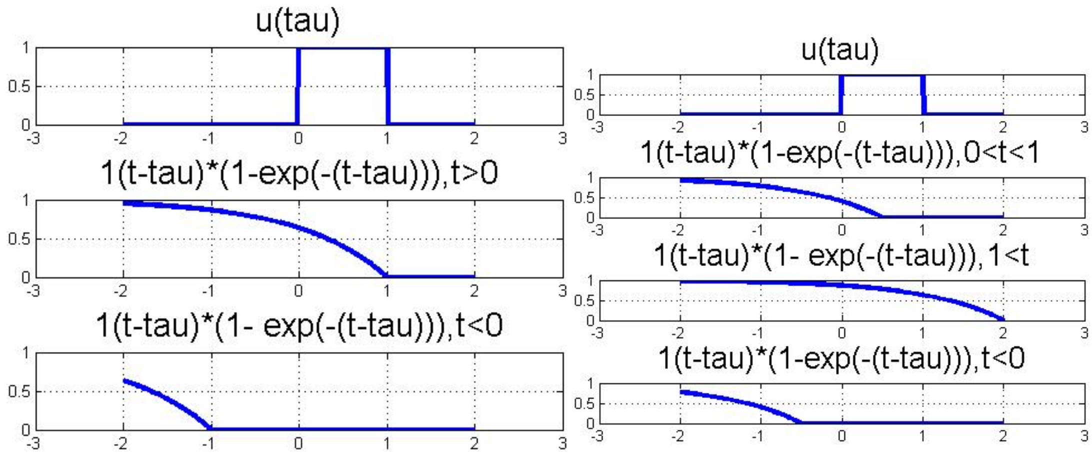
$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} u(\tau)1(t-\tau)(1 - e^{-(t-\tau)})d\tau$$

The "hard part" is the $u(\tau)1(t-\tau)$ on the integration range.

The following diagrams shows all the functions corresponds to different t







Refer to the diagram above, hence

- When $t < 0$, $u(\tau)$ and $1(t-\tau)$ disjoint.
- When $0 < t < 1$, $u(\tau)1(t-\tau)$ change integration range from $[-\infty \quad \infty]$ to $[0 \quad t]$.
- When $1 < t$, $u(\tau)1(t-\tau)$ change the integration range from $[-\infty \quad \infty]$ to $[0 \quad 1]$.

Hence

$$y(t) = \begin{cases} \int_{\tau=0}^{\tau=1} 1 - e^{-(t-\tau)} d\tau & t > 1 \\ \int_{\tau=0}^{\tau=t} 1 - e^{-(t-\tau)} d\tau & 0 < t < 1 \\ 0 & t < 0 \end{cases}$$

After integration,

$$y(t) = \begin{cases} 1 - (e-1)e^{-t} & t > 1 \\ t - e^{-t}(e^{t} - 1) & 0 < t < 1 \\ 0 & t < 0 \end{cases}$$

Which is

$$y(t) = \begin{cases} 1 - (e-1)e^{-t} & t > 1 \\ t - 1 + e^{-t} & 0 < t < 1 \\ 0 & t < 0 \end{cases}$$

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