Oblique Incidence of Plane Waves

May 16, 2013

1. The general problem
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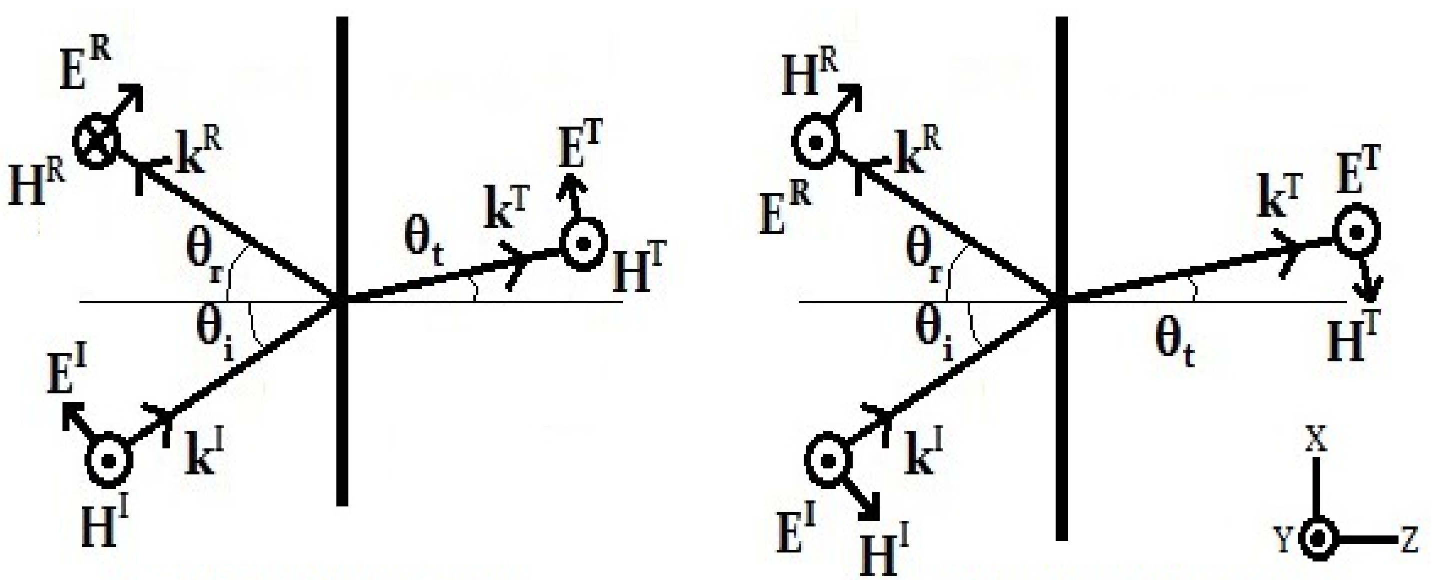
1 The general problem



Parallel Polarization
E-field on the paper

Perpendocular Polarization
E-field out of paper

In time domain

$$\begin{cases} \underline{E^I}\left(\underline{r},t\right) = E_0^I \cos\left(\omega t - \underline{k^I}\cdot\underline{r}\right)\hat{k}_E^I \\ \underline{E^R}\left(\underline{r},t\right) = E_0^R \cos\left(\omega t - \underline{k^R}\cdot\underline{r}\right)\hat{k}_R^R \\ \underline{E^T}\left(\underline{r},t\right) = E_0^T \cos\left(\omega t - \underline{k}^T\cdot\underline{r}\right)\hat{k}_E^T \end{cases} \qquad \begin{cases} \underline{H^I}\left(\underline{r},t\right) = H_0^I \cos\left(\omega t - \underline{k^I}\cdot\underline{r}\right)\hat{k}_H^I \\ \underline{H^R}\left(\underline{r},t\right) = H_0^R \cos\left(\omega t - \underline{k^R}\cdot\underline{r}\right)\hat{k}_H^R \\ \underline{H^T}\left(\underline{r},t\right) = H_0^T \cos\left(\omega t - \underline{k}^T\cdot\underline{r}\right)\hat{k}_H^T \end{cases}$$

Simplified in phasor domain

$$\begin{cases} \underline{E^I}\left(\underline{r}\right) = E_0^I \exp\left(-j\underline{k^I}\cdot\underline{r}\right)\hat{k}_E^I \\ \underline{E^R}\left(\underline{r}\right) = E_0^R \exp\left(-j\underline{k^R}\cdot\underline{r}\right)\hat{k}_R^R \\ \underline{E^T}\left(\underline{r}\right) = E_0^T \exp\left(-j\underline{k}^T\cdot\underline{r}\right)\hat{k}_E^T \end{cases} \qquad \begin{cases} \underline{H^I}\left(\underline{r}\right) = H_0^I \exp\left(-j\underline{k^I}\cdot\underline{r}\right)\hat{k}_H^I \\ \underline{H^R}\left(\underline{r}\right) = H_0^R \exp\left(-j\underline{k^R}\cdot\underline{r}\right)\hat{k}_H^R \\ \underline{H^T}\left(\underline{r}\right) = H_0^T \exp\left(-j\underline{k}^T\cdot\underline{r}\right)\hat{k}_H^T \end{cases}$$

The problem is generalized as follows

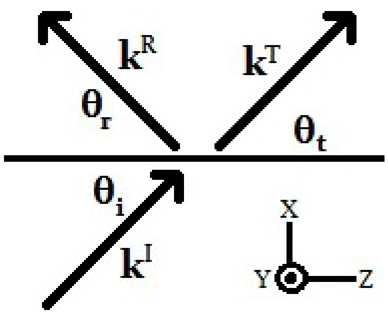
Given the following information

$$\begin{cases} \text{Incident wave information :} \begin{cases} E_0^I : \text{magnitude of incident wave} \\ \hat{k} : \text{direction of incident wave} \\ \omega : \text{frequency of the incident wave} \end{cases} \\ \text{Material medium information :} \begin{cases} \varepsilon : \text{permeativity} \\ \mu : \text{permeability} \\ \sigma : \text{condictivitiy} \end{cases} \end{cases}$$

To find out the following unknonwns

$$\begin{cases} \text{Magnitude of reflected, transmitted wave} \\ \text{Direction of reflected, transmitted wave} \end{cases}$$

2 The Wave vector $\underline{k}$



$$\begin{cases} \underline{k}^I : \text{the incidient wave vector} \\ \underline{k}^R : \text{the reflected wave vector} \\ \underline{k}^T : \text{the transmitted wave vector} \end{cases}$$

Recall, any vector $\underline{V} = \hat{V}|\underline{V}| = \hat{V}V$, thus

$$\begin{cases} k^I : \text{the magnitude of } \underline{k}^I \\ k^R : \text{the magnitude of } \underline{k}^R \\ k^T : \text{the magnitude of } \underline{k}^T \end{cases} \quad \begin{cases} \hat{k}^I : \text{the direction of} \underline{k}^I \\ \hat{k}^R : \text{the direction of} \underline{k}^R \\ \hat{k}^T : \text{the direction of} \underline{k}^T \end{cases} \quad \text{and } \underline{k} = \hat{k}k$$

Furthermore

$$\begin{cases} k_1 : \text{the magnitude of wave vector in media 1} \\ k_2 : \text{the magnitude of wave vector in media 2} \end{cases} \quad where \;\; k_i = \omega\sqrt{\mu_i\varepsilon_i}$$

Thus

$$k^I = k^R = k_1 = \omega\sqrt{\mu_1\varepsilon_1} \qquad k^T = k_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

And recall, vector can be decompose into it's basis vector

$$\begin{cases} \underline{k}^I = k_{Ix}\hat{x} + k_{Iz}\hat{z} = k^I\sin\theta_i\hat{x} + k^I\cos\theta_i\hat{z} \\ \underline{k}^R = k_{Rx}\hat{x} - k_{Rz}\hat{z} = k^R\sin\theta_r\hat{x} - k^R\cos\theta_r\hat{z} \\ \underline{k}^T = k_{Tx}\hat{x} + k_{Tz}\hat{z} = k^T\sin\theta_T\hat{x} + k^T\cos\theta_T\hat{z} \end{cases}$$

Thus

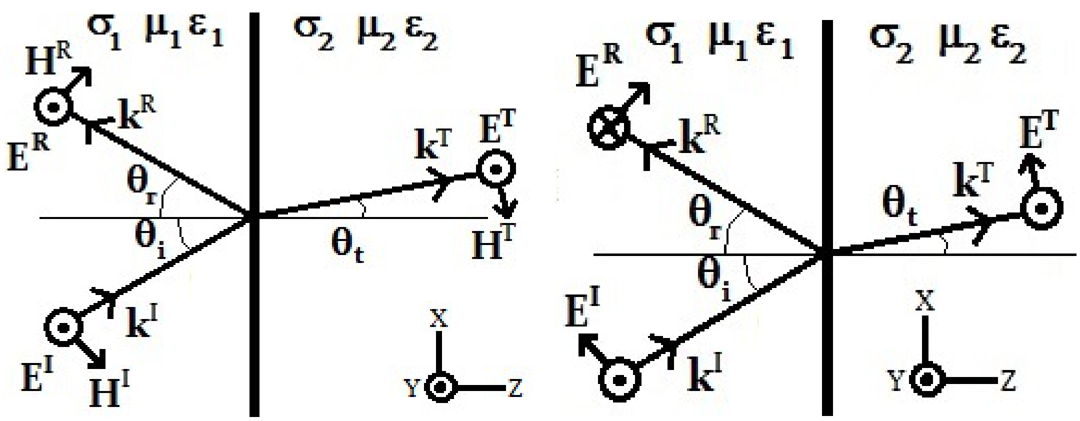
$$\begin{cases} \hat{k}^I = \frac{\underline{k}^I}{|\underline{k}^I|} = \frac{k_{Ix}\hat{x} + k_{Iz}\hat{z}}{k^I} = \sin\theta_i\hat{x} + \cos\theta_i\hat{z} \\ \hat{k}^R = \frac{\underline{k}^R}{|\underline{k}^R|} = \frac{k_{Rx}\hat{x} - k_{Rz}\hat{z}}{k^R} = \sin\theta_r\hat{x} - \cos\theta_r\hat{z} \\ \hat{k}^T = \frac{\underline{k}^T}{|\underline{k}^T|} = \frac{k_{Tx}\hat{x} + k_{Tz}\hat{z}}{k^T} = \sin\theta_T\hat{x} + \cos\theta_T\hat{z} \end{cases}$$

And

$$\begin{cases} \underline{k}^I \cdot \underline{r} = k_{Ix}x + k_{Iz}z = k^I\left(\sin\theta_i x + \cos\theta_i z\right) \\ \underline{k}^R \cdot \underline{r} = k_{Rx}x - k_{Rz}z = k^R\left(\sin\theta_r x - \cos\theta_r z\right) \\ \underline{k}^T \cdot \underline{r} = k_{Tx}x + k_{Tz}z = k^T\left(\sin\theta_T x + \cos\theta_T z\right) \end{cases}$$

3 Perpendicular Polarizations

- Expression of $E^{I,R,T}$and $H^{I,R,T}$ and their direction term, propagation term
- Fresnel Relationship and Fresnel Coefficient
- Solving Fresnel Equations
- Special case : Normal incident and only one media



By looking at the diagram

$$\begin{cases} \text{Perpendicular case : all direction of } E \text{ are } +\hat{y}\text{ , so } \hat{k}_E^{I,R,T} = +\hat{y} \\ \text{Parallel case : direction of } H \text{ are} \pm \hat{y}, \text{ so } \hat{k}_H^{I,T} = +\hat{y}\,, \hat{k}_H^{R} = -\hat{y} \end{cases}$$

Thus

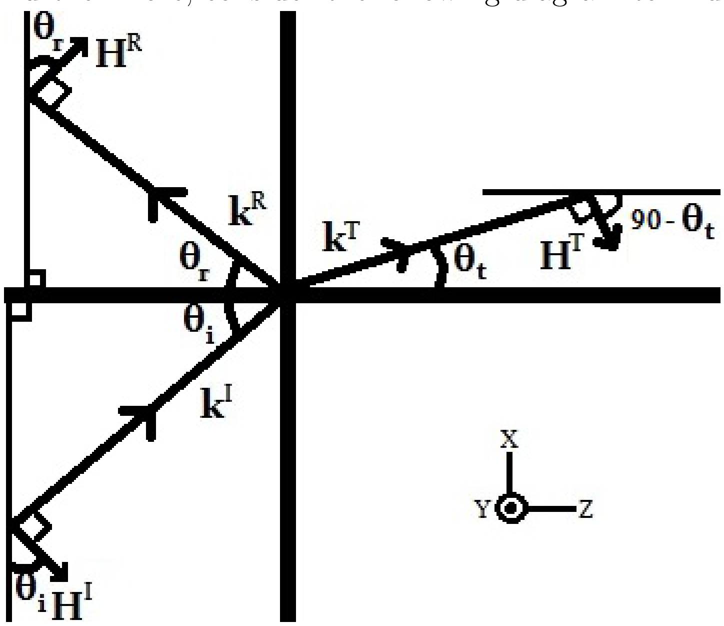
$$\perp \begin{cases} \underline{E^I}\left(\underline{r}\right) = E_0^I \exp j\left(-k_{Ix}x - k_{Iz}z\right)\hat{y} & \underline{H^I}\left(\underline{r}\right) = H_0^I \exp j\left(-k_{Ix}\hat{x} - k_{Iz}\hat{z}\right)\left(\hat{k}_H^I\right) \\ \underline{E^R}\left(\underline{r}\right) = E_0^R \exp j\left(-k_{Rx}x + k_{Rz}z\right)\hat{y} & \underline{H^R}\left(\underline{r}\right) = H_0^R \exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(\hat{k}_H^R\right) \\ \underline{E^T}\left(\underline{r}\right) = E_0^T \exp j\left(-k_{Tx}x - k_{Tz}z\right)\hat{y} & \underline{H^T}\left(\underline{r}\right) = H_0^T \exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(\hat{k}_H^T\right) \end{cases}$$

$$|| \begin{cases} \underline{E^I}\left(\underline{r}\right) = E_0^I \exp j\left(-k_{Ix}x - k_{Iz}z\right)\left(\hat{k}_E^I\right) & \underline{H^I}\left(\underline{r}\right) = H_0^I \exp j\left(-k_{Ix}\hat{x} - k_{Iz}\hat{z}\right)\left(\hat{y}\right) \\ \underline{E^R}\left(\underline{r}\right) = E_0^R \exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(\hat{k}_E^R\right) & \underline{H^R}\left(\underline{r}\right) = H_0^R \exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(-\hat{y}\right) \\ \underline{E^T}\left(\underline{r}\right) = E_0^T \exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(\hat{k}_E^T\right) & \underline{H^T}\left(\underline{r}\right) = H_0^T \exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(\hat{y}\right) \end{cases}$$

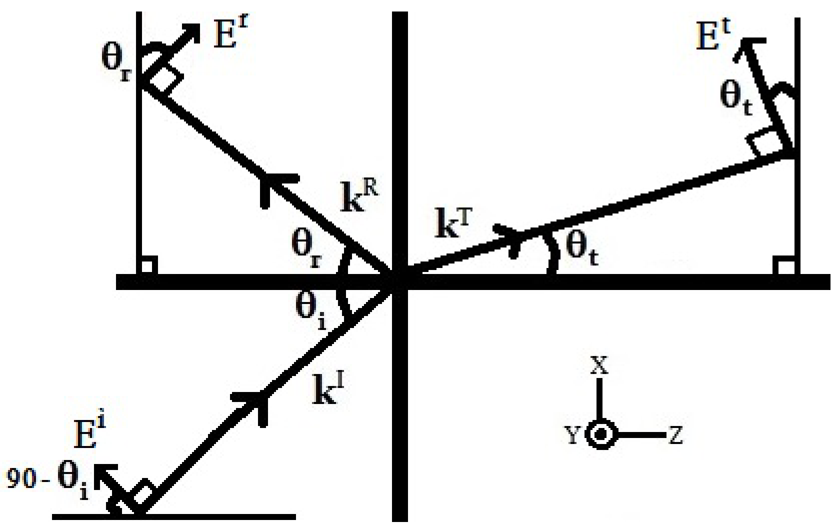
Apply Fresnel Relationship and Relationship between H_0 and E_0

$$\begin{cases} E_0^R = R_E E_0^I \\ E_0^T = T_E E_0^I \end{cases} \quad \text{and } H_0 = \frac{E_0}{\eta} \quad \Longrightarrow \begin{cases} E_0^I & H_0^I = \dfrac{E_0^I}{\eta_1} \\ E_0^R = RE_0^I & H_0^R = \dfrac{RE_0^I}{\eta_1} \\ E_0^T = TE_0^I & H_0^T = \dfrac{TE_0^I}{\eta_2} \end{cases}$$

Further more, consider the following diagram to find direction of H ($\perp$ case) and E ($||$ case)



$$\begin{cases} \hat{k}_H^I = -\cos\theta_i \hat{x} + \sin\theta_i \hat{z} \\ \hat{k}_H^R = \cos\theta_r \hat{x} + \sin\theta_r \hat{z} \\ \hat{k}_H^T = -\sin\theta_T \hat{x} + \cos\theta_T \hat{z} \end{cases}$$



$$\begin{cases} \hat{k}_E^I = \cos\theta_i \hat{x} - \sin\theta_i \hat{z} \\ \hat{k}_E^R = \cos\theta_r \hat{x} + \sin\theta_r \hat{z} \\ \hat{k}_E^T = \cos\theta_T \hat{x} - \sin\theta_T \hat{z} \end{cases}$$

Therefore

$$\perp \begin{cases} \underline{E^I}(\underline{r}) = E_0^I \exp j\left(-k_{Ix}x - k_{Iz}z\right)\hat{y} & \underline{H^I}(\underline{r}) = \dfrac{E_0^I}{\eta_1}\exp j\left(-k_{Ix}\hat{x} - k_{Iz}\hat{z}\right)\left(-\cos\theta_i\hat{x} + \sin\theta_i\hat{z}\right) \\ \underline{E^R}(\underline{r}) = R_{\perp E}E_0^I \exp j\left(-k_{Rx}x + k_{Rz}z\right)\hat{y} & \underline{H^R}(\underline{r}) = \dfrac{R_{\perp E}E_0^I}{\eta_1}\exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right) \\ \underline{E^T}(\underline{r}) = T_{\perp E}E_0^I \exp j\left(-k_{Tx}x - k_{Tz}z\right)\hat{y} & \underline{H^T}(\underline{r}) = \dfrac{T_{\perp E}E_0^I}{\eta_2}\exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(-\sin\theta_T\hat{x} + \cos\theta_T\hat{z}\right) \end{cases}$$

$$|| \begin{cases} \underline{E^I}(\underline{r}) = E_0^I \exp j\left(-k_{Ix}x - k_{Iz}z\right)\left(\cos\theta_i\hat{x} - \sin\theta_i\hat{z}\right) & \underline{H^I}(\underline{r}) = \dfrac{E_0^I}{\eta_1}\exp j\left(-k_{Ix}\hat{x} - k_{Iz}\hat{z}\right)\left(\hat{y}\right) \\ \underline{E^R}(\underline{r}) = R_{||E}E_0^I \exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right) & \underline{H^R}(\underline{r}) = \dfrac{R_{||E}E_0^I}{\eta_1}\exp j\left(-k_{Rx}x + k_{Rz}z\right)\left(-\hat{y}\right) \\ \underline{E^T}(\underline{r}) = T_{||E}E_0^I \exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(\cos\theta_T\hat{x} - \sin\theta_T\hat{z}\right) & \underline{H^T}(\underline{r}) = \dfrac{T_{||E}E_0^I}{\eta_2}\exp j\left(-k_{Tx}x - k_{Tz}z\right)\left(\hat{y}\right) \end{cases}$$

Now the remaining works to do is to find those Fresnel Coefficient : R and T

Those value can be found by using boundary condition : basically, the term e^{-jkz} disappear when puting $z = 0$

$$\perp\,(z=0) \begin{cases} \underline{E^I}(x) = E_0^I \exp j\left(-k_{Ix}x\right)\hat{y} & \underline{H^I}(x) = \dfrac{E_0^I}{\eta_1}\exp j\left(-k_{Ix}\hat{x}\right)\left(-\cos\theta_i\hat{x} + \sin\theta_i\hat{z}\right) \\ \underline{E^R}(x) = R_{\perp E}E_0^I \exp j\left(-k_{Rx}x\right)\hat{y} & \underline{H^R}(x) = \dfrac{R_{\perp E}E_0^I}{\eta_1}\exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right) \\ \underline{E^T}(x) = T_{\perp E}E_0^I \exp j\left(-k_{Tx}x\right)\hat{y} & \underline{H^T}(x) = \dfrac{T_{\perp E}E_0^I}{\eta_2}\exp j\left(-k_{Tx}x\right)\left(-\sin\theta_T\hat{x} + \cos\theta_T\hat{z}\right) \end{cases}$$

$$||\,(z=0) \begin{cases} \underline{E^I}(x) = E_0^I \exp j\left(-k_{Ix}x\right)\left(\cos\theta_i\hat{x} - \sin\theta_i\hat{z}\right) & \underline{H^I}(x) = \dfrac{E_0^I}{\eta_1}\exp j\left(-k_{Ix}\hat{x}\right)\left(\hat{y}\right) \\ \underline{E^R}(x) = R_{||E}E_0^I \exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right) & \underline{H^R}(x) = \dfrac{R_{||E}E_0^I}{\eta_1}\exp j\left(-k_{Rx}x\right)\left(-\hat{y}\right) \\ \underline{E^T}(x) = T_{||E}E_0^I \exp j\left(-k_{Tx}x\right)\left(\cos\theta_T\hat{x} - \sin\theta_T\hat{z}\right) & \underline{H^T}(x) = \dfrac{T_{||E}E_0^I}{\eta_2}\exp j\left(-k_{Tx}x\right)\left(\hat{y}\right) \end{cases}$$

Apply boundary condition

"Tangential component of E fields are equal"

"Tangential component of H fields are equal"

Note , Generally $H_{t1} - H_{t2} = J_s$, but this is the case in equilibrium

For the boundary at z , the tangential directions are x and y

That is

$$\perp \begin{cases} \text{y-component of } \left(E^I + E^R\right)_{\text{at z=0}} = \text{y-component of } \left(E^T\right)_{\text{at z=0}} \\ \\ \text{x-component of } \left(H^I + H^R\right)_{\text{at z=0}} = \text{x-component of } \left(H^T\right)_{\text{at z=0}} \end{cases}$$

$$|| \left\{ \begin{array}{c} \text{x-component of } \left(E^I + E^R\right)_{\text{at z=0}} = \text{x-component of } \left(E^T\right)_{\text{at z=0}} \\ \\ \text{y-component of } \left(H^I + H^R\right)_{\text{at z=0}} = \text{xy-component of } \left(H^T\right)_{\text{at z=0}} \end{array} \right.$$

Which is

$$\perp (z=0) \left\{ \begin{array}{cc} \underline{E^I}(x) = E_0^I \exp j\left(-k_{Ix}x\right) & \underline{H^I}(x) = \dfrac{E_0^I}{\eta_1} \exp j\left(-k_{Ix}\hat{x}\right)\left(-\cos\theta_i\right) \\ + & + \\ \underline{E^R}(x) = R_{\perp E} E_0^I \exp j\left(-k_{Rx}x\right) & \underline{H^R}(x) = \dfrac{R_{\perp E} E_0^I}{\eta_1} \exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right) \\ \| & \| \\ \underline{E^T}(x) = T_{\perp E} E_0^I \exp j\left(-k_{Tx}x\right) & \underline{H^T}(x) = \dfrac{T_{\perp E} E_0^I}{\eta_2} \exp j\left(-k_{Tx}x\right)\left(-\cos\theta_T\right) \end{array} \right.$$

$$|| (z=0) \left\{ \begin{array}{cc} \underline{E^I}(x) = E_0^I \exp j\left(-k_{Ix}x\right)\left(\cos\theta_i\right) & \underline{H^I}(x) = \dfrac{E_0^I}{\eta_1} \exp j\left(-k_{Ix}\hat{x}\right) \\ + & + \\ \underline{E^R}(x) = R_{||E} E_0^I \exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right) & \underline{H^R}(x) = \dfrac{R_{||E} E_0^I}{\eta_1} \exp j\left(-k_{Rx}x\right)(-1) \\ \| & \| \\ \underline{E^T}(x) = T_{||E} E_0^I \exp j\left(-k_{Tx}x\right)\left(\cos\theta_T\right) & \underline{H^T}(x) = \dfrac{T_{||E} E_0^I}{\eta_2} \exp j\left(-k_{Tx}x\right) \end{array} \right.$$

i.e.

$$\perp \left\{ \begin{array}{cc} E_0^I \exp j\left(-k_{Ix}x\right) & \dfrac{E_0^I}{\eta_1} \exp j\left(-k_{Ix}\hat{x}\right)\left(-\cos\theta_i\right) \\ + & + \\ R_{\perp E} E_0^I \exp j\left(-k_{Rx}x\right) & \dfrac{R_{\perp E} E_0^I}{\eta_1} \exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right) \\ \| & \| \\ T_{\perp E} E_0^I \exp j\left(-k_{Tx}x\right) & \dfrac{T_{\perp E} E_0^I}{\eta_2} \exp j\left(-k_{Tx}x\right)\left(-\cos\theta_T\right) \end{array} \right.$$

$$|| \left\{ \begin{array}{cc} E_0^I \exp j\left(-k_{Ix}x\right)\left(\cos\theta_i\right) & \dfrac{E_0^I}{\eta_1} \exp j\left(-k_{Ix}\hat{x}\right) \\ + & + \\ R_{||E} E_0^I \exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right) & \dfrac{R_{||E} E_0^I}{\eta_1} \exp j\left(-k_{Rx}x\right)(-1) \\ \| & \| \\ T_{||E} E_0^I \exp j\left(-k_{Tx}x\right)\left(\cos\theta_T\right) & \dfrac{T_{||E} E_0^I}{\eta_2} \exp j\left(-k_{Tx}x\right) \end{array} \right.$$

Rearrange

$$\perp\left\{\begin{array}{c}\exp j\left(-k_{Ix}x\right)+R_{\perp E}\exp j\left(-k_{Rx}x\right)=T_{\perp E}\exp j\left(-k_{Tx}x\right)\\ \\ \frac{1}{\eta_1}\left[\exp j\left(-k_{Ix}\hat{x}\right)\left(-\cos\theta_i\right)+R_{\perp E}\exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right)\right]=\frac{T_{\perp E}}{\eta_2}\exp j\left(-k_{Tx}x\right)\left(-\cos\theta_T\right)\end{array}\right.$$

$$||\left\{\begin{array}{c}\exp j\left(-k_{Ix}x\right)\left(\cos\theta_i\right)+R_{||E}\exp j\left(-k_{Rx}x\right)\left(\cos\theta_r\right)=T_{||E}\exp j\left(-k_{Tx}x\right)\left(\cos\theta_T\right)\\ \\ \frac{1}{\eta_1}\left[\exp j\left(-k_{Ix}\hat{x}\right)-R_{||E}\exp j\left(-k_{Rx}x\right)\right]=\frac{T_{||E}}{\eta_2}\exp j\left(-k_{Tx}x\right)\end{array}\right.$$

Since the boundary condition should hold for all x , then

$$k_{Ix}=k_{Rx}=k_{Tx}\triangleq k_x \quad (1)$$

i.e. They are Law of reflection and refraction

$$\left\{\begin{array}{c}k_{Ix}=k_{Rx}\iff k_1\sin\theta_i=k_1\sin\theta_r\iff\theta_i=\theta_r\\ k_{Ix}=k_{Tx}\iff k_1\sin\theta_i=k_2\sin\theta_2\end{array}\right.$$

Then since

$$\left\{\begin{array}{c}\left(k^I\right)^2=k_{Ix}^2+k_{Iz}^2=k_1^2=\omega^2\mu_1\varepsilon_1\\ \left(k^R\right)^2=k_{Rx}^2+k_{Rz}^2=k_1^2=\omega^2\mu_1\varepsilon_1\\ \left(k^T\right)^2=k_{Tx}^2+k_{Tz}^2=k_2^2=\omega^2\mu_2\varepsilon_2\end{array}\right.$$

Therefore

$$\left\{\begin{array}{c}k_{Iz}=\sqrt{k_1^2-k_{Ix}^2}=\sqrt{k_1^2-k_x^2}=\sqrt{k_1^2-k_{Rx}^2}=k_{Rz}\\ k_{Tz}=\sqrt{k_2^2-k_{Tx}^2}=\sqrt{k_2^2-k_x^2}\end{array}\right.$$

Then

$$k_{Iz}=k_{Rz}\triangleq k_z \quad (2)$$

Therefore the Boundary Condition Equation can be simplified (factor out common term e^{-jk_xx})

$$\perp\left\{\begin{array}{c}1+R_{\perp E}=T_{\perp E}\\ \frac{1}{\eta_1}\left[-\cos\theta_i+R_{\perp E}\cos\theta_r\right]=\frac{T_{\perp E}}{\eta_2}\left(-\cos\theta_T\right)\end{array}\right.$$

$$||\left\{\begin{array}{c}\cos\theta_i+R_{||E}\cos\theta_r=T_{||E}\cos\theta_T\\ \frac{1}{\eta_1}\left[1-R_{||E}\right]=\frac{T_{||E}}{\eta_2}\end{array}\right.$$

And since $\theta_i=\theta_r$ (Law of reflection) , rearrange same term together

$$\perp\left\{\begin{array}{c}1+R_{\perp E}=T_{\perp E}\\ 1-R_{\perp E}=\frac{\eta_1}{\eta_2}\frac{\cos\theta_T}{\cos\theta_i}T_{\perp E}\end{array}\right. \qquad ||\left\{\begin{array}{c}1+R_{||E}=\frac{\cos\theta_T}{\cos\theta_i}T_{||E}\\ 1-R_{||E}=\frac{\eta_1}{\eta_2}T_{||E}\end{array}\right.$$

Solving, the Fresnel Coefficients are

$$\perp \begin{cases} T_{\perp E} = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_T + \eta_2 \cos\theta_i} \\ R_{\perp E} = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_T}{\eta_1 \cos\theta_T + \eta_2 \cos\theta_i} \end{cases} \qquad || \begin{cases} T_{||E} = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_T} \\ R_{||E} = \dfrac{\eta_2 \cos\theta_T - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_T} \end{cases}$$

4 Special cases

4.1 Normal incidence

Normal incident : When $\theta_i = 0 = \theta_T$

$$\begin{aligned} T_{\perp E} = T_{||E} &= \frac{2\eta_2}{\eta_1 + \eta_2} \quad \text{There are the same} \\ R_{\perp E} = R_{||E} &= \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} \end{aligned}$$

Further, when $\eta_2 = \eta_1$ (only one media)

$$\begin{aligned} T_{\perp E} = T_{||E} = 1 \text{ (all transmitted)} \\ R_{\perp E} = R_{||E} = 0 \text{ (No reflection)} \end{aligned}$$

4.2 Berewster Angle

$$\perp \begin{cases} T_{\perp E} = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_T + \eta_2 \cos\theta_i} \\ R_{\perp E} = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_T}{\eta_1 \cos\theta_T + \eta_2 \cos\theta_i} \end{cases} \qquad || \begin{cases} T_{||E} = \dfrac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_T} \\ R_{||E} = \dfrac{\eta_2 \cos\theta_T - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_T} \end{cases}$$

Notice that it is possible for $R_\perp = 0$ and $R_{||} = 0$when

$$\eta_2 \cos\theta_i = \eta_1 \cos\theta_T \text{ for } R_\perp$$

$$\eta_2 \cos\theta_T = \eta_1 \cos\theta_i \text{ for } R_{||}$$

Do the following (1.turn cos into sin term , 2. Apply Law of refraction to sin$\theta_T = \dfrac{k_1}{k_2} \sin\theta_i$)

$$\eta_2^2 \left(1 - \sin^2\theta_i\right) = \eta_1^2 \left(1 - \frac{k_1^2}{k_2^2} \sin^2\theta_i\right) \qquad \text{for } R_\perp$$

$$\eta_2^2 \left(1 - \frac{k_1^2}{k_2^2} \sin^2\theta_i\right) = \eta_1^2 \left(1 - \sin_i^2\theta\right) \qquad \text{for } R_{||}$$

i.e.

$$\sin\theta_i = \sqrt{\frac{\eta_1^2 - \eta_2^2}{\eta_1^2 \frac{k_1^2}{k_2^2} - \eta_2^2}} \qquad \text{for } R_\perp$$

$$\sin\theta_i = \sqrt{\frac{\eta_1^2 - \eta_2^2}{\eta_1^2 - \eta_2^2 \frac{k_1^2}{k_2^2}}} \qquad \text{for } R_{||}$$

Expand k , η

For $R_\perp$

$$\sin\theta_i = \sqrt{\frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1}\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2}{\varepsilon_2}}} = \sqrt{\frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1}\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2}{\varepsilon_2}} \times \frac{\frac{\varepsilon_2}{\mu_1}}{\frac{\varepsilon_2}{\mu_1}}}$$

$$\sin\theta_i = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\mu_1}{\mu_2} - \frac{\mu_2}{\mu_1}}}$$

For $R_{||}$

$$\sin\theta_i = \sqrt{\frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}} = \sqrt{\frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} \times \frac{\frac{\varepsilon_2}{\mu_1}}{\frac{\varepsilon_2}{\mu_1}}}$$

$$\sin\theta_i = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}}}$$

Therefore, denote this angle as Brewster Angle θ_B

$$\sin\theta_{B\perp} = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\mu_1}{\mu_2} - \frac{\mu_2}{\mu_1}}} \qquad \sin\theta_{B||} = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\mu_2}{\mu_1}}{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}}}$$

A special case , when $\mu_1 = \mu_2 = \mu_0$ (non-magnetic material)

$$\sin\theta_{B\perp} = \infty(\text{impossible}) \qquad \sin\theta_{B||} = \sqrt{\frac{\frac{\varepsilon_2}{\varepsilon_1} - 1}{\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_1}{\varepsilon_2}}} = \sqrt{\frac{\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}}{\frac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_1\varepsilon_2}}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$$

By $1 + \tan^2\theta = \sec^2\theta$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{1}{\cos^2\theta} - 1} = \sqrt{\frac{1}{1 - \sin^2\theta} - 1}$$

Plug in $\sin\theta_{B||}$

$$\tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

i.e. It is impossible for $R_\perp = 0$ for non-magnetic material, but possible for $R_{||}$

–END–