The General Derivation of Waveguide

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General Solution

Consider Maxwell's Equations (in phasor form)

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$$

By $\nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x(x,y,z) & F_y(x,y,z) & F_z(x,y,z) \end{vmatrix}$

Then

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x,y,z) & E_y(x,y,z) & E_z(x,y,z) \end{vmatrix} = -\mu \left(\frac{\partial H_x}{\partial t}, \frac{\partial H_y}{\partial t}, \frac{\partial H_z}{\partial t} \right)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x,y,z) & E_y(x,y,z) & E_z(x,y,z) \end{vmatrix} = -\varepsilon \left(\frac{\partial E_x}{\partial t}, \frac{\partial E_y}{\partial t}, \frac{\partial E_z}{\partial t} \right)$$

Equate x, y, z component

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \qquad -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon \frac{\partial E_z}{\partial t}$$

To simplify the mathematics, we can assume the solution of the fields are in the following form (consider propagating along z)

$$E(x,y,z) = E(x,y)e^{-j\beta z} = (\mathbf{e}(x,y) + e_z(x,y)\hat{z})\, e^{-j\beta z}$$

$$H(x,y,z) = H(x,y)e^{-j\beta z} = (\mathbf{h}(x,y) + h_z(x,y)\hat{z})\, e^{-j\beta z}$$

Thus

$$\frac{\partial}{\partial z} = -j\beta$$

In form means that the field propagation term $e^{-j\beta z}$ is along the z direction , and the $f(x,y)$ term is the amplitude term
Recall that propgation term $e^{-j\beta z}$ becomes familiar $\cos(\omega t - \beta z)$ or $\sin(\omega t - \beta z)$ after phasor transform

$$f(x,y,z,t) = \mathbf{Re}\left[e^{j\omega t} f(x,y,z)\right]$$

Since the fields (in phasor) is assumed to be time harmonic

$$\frac{\partial}{\partial t} = j\omega$$

The Maxwell's Equations become

$$H_x = \frac{\dfrac{\partial E_z}{\partial y} + j\beta E_y}{-j\omega\mu} \quad (1) \qquad E_x = \frac{\dfrac{\partial H_z}{\partial y} + j\beta H_y}{j\omega\varepsilon} \quad (4)$$

$$H_y = \frac{-\dfrac{\partial E_z}{\partial x} - j\beta E_x}{-j\omega\mu} \quad (2) \qquad E_y = \frac{-\dfrac{\partial H_z}{\partial x} - j\beta H_x}{j\omega\varepsilon} \quad (5)$$

$$H_z = \frac{\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}}{-j\omega\mu} \quad (3) \qquad E_z = \frac{\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y}}{j\omega\varepsilon} \quad (6)$$

Express other fields as E_z , H_z
Put (5) $\to$ (1)

$$H_x = \frac{\dfrac{\partial E_z}{\partial y} + j\beta\left(\dfrac{-\dfrac{\partial H_z}{\partial x} - j\beta H_x}{j\omega\varepsilon}\right)}{-j\omega\mu}$$

$$= -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x} - \frac{j\beta}{\omega^2\mu\varepsilon}j\beta H_x$$

$$= \frac{j}{\omega\mu}\frac{\partial E_z}{\partial y} - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x} + \frac{\beta^2}{\omega^2\mu\varepsilon}H_x$$

$$\Longleftrightarrow \qquad \left(1 - \frac{\beta^2}{\omega^2\mu\varepsilon}\right)H_x = \frac{j}{\omega\mu}\frac{\partial E_z}{\partial y} - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x}$$

$$\Longleftrightarrow \qquad H_x = \frac{\frac{j}{\omega\mu}}{\left(1 - \dfrac{\beta^2}{\omega^2\mu\varepsilon}\right)}\left(\frac{\partial E_z}{\partial y} - \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial x}\right)$$

$$\Longleftrightarrow \qquad H_x = \frac{j\omega\varepsilon}{(\omega^2\mu\varepsilon - \beta^2)}\left(\frac{\partial E_z}{\partial y} - \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial x}\right)$$

$$\Longleftrightarrow \qquad H_x = \frac{j}{(\omega^2\mu\varepsilon - \beta^2)}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

Let $k = \omega\sqrt{\mu\varepsilon}$, and define cutoff wave number

$$k_c^2 = k^2 - \beta^2$$

$$\Longleftrightarrow \qquad H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

Consider the equation

$$H_x = \frac{\dfrac{\partial E_z}{\partial y} + j\beta E_y}{-j\omega\mu} \quad (1) \qquad E_x = \frac{\dfrac{\partial H_z}{\partial y} + j\beta H_y}{j\omega\varepsilon} \quad (4)$$

$$H_y = \frac{-\dfrac{\partial E_z}{\partial x} - j\beta E_x}{-j\omega\mu} \quad (2) \qquad E_y = \frac{-\dfrac{\partial H_z}{\partial x} - j\beta H_x}{j\omega\varepsilon} \quad (5)$$

(4) $\to$ (2) , (2) $\to$ (4) , (1) $\to$ (5) and do the same procedure as above, we can obtain the following fields expressions

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \qquad E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) \qquad E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$$

Solution of $z-$fields

Consider the systems of Maxwell's Equations form general waveguide

$$H_x = \frac{\frac{\partial E_z}{\partial y} + j\beta E_y}{-j\omega\mu} \quad (1) \qquad E_x = \frac{\frac{\partial H_z}{\partial y} + j\beta H_y}{j\omega\varepsilon} \quad (4)$$

$$H_y = \frac{-\frac{\partial E_z}{\partial x} - j\beta E_x}{-j\omega\mu} \quad (2) \qquad E_y = \frac{-\frac{\partial H_z}{\partial x} - j\beta H_x}{j\omega\varepsilon} \quad (5)$$

$$H_z = \frac{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}}{-j\omega\mu} \quad (3) \qquad E_z = \frac{\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}}{j\omega\varepsilon} \quad (6)$$

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \quad (7) \qquad E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) \quad (9)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) \quad (8) \qquad E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) \quad (10)$$

(7), (8) →(6)

$$E_z = \frac{\frac{\partial}{\partial x}\left(\frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)\right)}{j\omega\varepsilon}$$

$\Longleftrightarrow$

$$k_c^2 E_z = -\frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2}$$

$\Longleftrightarrow$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0 \qquad \Longleftrightarrow \qquad \nabla^2 E_z + k_c^2 E_z = 0$$

In the same way ,

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_c^2 H_z = 0 \qquad \Longleftrightarrow \qquad \nabla^2 H_z + k_c^2 H_z = 0$$

i.e.

$$\begin{cases} \nabla^2 E_z + k_c^2 E_z = 0 \\ \nabla^2 H_z + k_c^2 H_z = 0 \end{cases}$$

After solving for z-fields, the x,y fields can be solved.

TEM, TE and TM

Apart form solving the fields from z-component, sometime there are some conditions

When $E_z = H_z = 0$, it is called TEM mode (Transverse Electromagnetic Wave), there is no fields in the direction of the propagation, i.e. Plane Wave

When $E_z = 0$, $H_z \neq 0$, it is called TE mode (Transverse Electric Wave), there is no E-fields in the direction of the propagation

When $E_z \neq 0$, $H_z = 0$, it is called TM mode (Transverse Magnetic Wave), there is no H-fields in the direction of the propagation

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