Phasor Form of General Waveguide Derivation

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For time-harmonic fields

$$\nabla \times \widetilde{E} = -j\omega\mu\widetilde{H} \qquad \nabla \times \widetilde{H} = j\omega\varepsilon\widetilde{E}$$

Form the study of EM wave propagation in free space, the 2 equations above can derive the following Helmholtz wave equation

$$\nabla^2 \left\{ \begin{array}{c} \widetilde{E} \\ \widetilde{H} \end{array} \right\} + \gamma^2 \left\{ \begin{array}{c} \widetilde{E} \\ \widetilde{H} \end{array} \right\} = 0$$

where $\gamma = \alpha + j\beta$

Assume the wave is propagate in z direction, thus the propagation term is characterized by $e^{-j\gamma z}$, and thus the fields can be expressed as

$$\widetilde{E}(x,y,z) = \tilde{e}(x,y)e^{-\gamma z} \qquad \widetilde{H}(x,y,z) = \tilde{h}(x,y)e^{-\gamma z}$$

where $e(x,y)$, $h(x,y)$ are the amplitude term, the term generally contain both tranverse field component and longitudinal field component

Therefore

$$\frac{\partial}{\partial z} \left\{ \begin{array}{c} \widetilde{E} \\ \widetilde{H} \end{array} \right\} = -\gamma \left\{ \begin{array}{c} \widetilde{E} \\ \widetilde{H} \end{array} \right\}$$

Or

$$\frac{\partial}{\partial z} = -\gamma$$

Expand the curl

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \widetilde{H}_X & \widetilde{H}_Y & \widetilde{H}_Z \end{vmatrix} = j\omega\varepsilon\widetilde{E} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & -\gamma \\ \widetilde{E}_X & \widetilde{E}_Y & \widetilde{E}_Z \end{vmatrix} = -j\omega\mu\widetilde{H}$$

Equate x, y, z component

$$\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon\widetilde{E}_X \quad (1) \qquad \frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu\widetilde{H}_X \quad (4)$$

$$-\frac{\partial H_z}{\partial x} - \gamma H_x = j\omega\varepsilon\widetilde{E}_Y \quad (2) \qquad -\frac{\partial E_z}{\partial x} - \gamma E_x = -j\omega\mu\widetilde{H}_Y \quad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon\widetilde{E}_Z \quad (3) \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu\widetilde{H}_Z \quad (6)$$

(1)(5)

$$\frac{\partial H_z}{\partial y} + \gamma \frac{-\frac{\partial E_z}{\partial x} - \gamma E_x}{-j\omega\mu} = j\omega\varepsilon \widetilde{E}_X$$

$$\frac{\partial H_z}{\partial y} + \gamma \frac{\frac{\partial E_z}{\partial x}}{j\omega\mu} = \left(j\omega\varepsilon - \frac{\gamma^2}{j\omega\mu} \right) \widetilde{E}_X = \left(\frac{\omega^2 \varepsilon\mu + \gamma^2}{-j\omega\mu} \right) \widetilde{E}_X$$

Let $h^2 = \gamma^2 + \omega^2 \mu\varepsilon = \gamma^2 + k^2$, or $\gamma = \sqrt{h^2 - k^2}$

$$\widetilde{E}_X = \frac{1}{h^2} \left(-j\omega\mu \frac{\partial H_z}{\partial y} - \gamma \frac{\partial E_z}{\partial x} \right)$$

Using same logic

$$\widetilde{H}_Y = \frac{1}{h^2} \left(-j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial y} \right)$$

(2)(4)

$$\widetilde{E}_Y = \frac{1}{h^2} \left(j\omega\mu \frac{\partial H_z}{\partial y} - \gamma \frac{\partial E_z}{\partial y} \right) \qquad \widetilde{H}_X = \frac{1}{h^2} \left(j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x} \right)$$

i.e.

$$\widetilde{E}_X = \frac{1}{h^2} \left(-j\omega\mu \frac{\partial H_z}{\partial y} - \gamma \frac{\partial E_z}{\partial x} \right) \qquad \widetilde{H}_X = \frac{1}{h^2} \left(j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x} \right)$$

$$\widetilde{E}_Y = \frac{1}{h^2} \left(j\omega\mu \frac{\partial H_z}{\partial y} - \gamma \frac{\partial E_z}{\partial y} \right) \qquad \widetilde{H}_Y = \frac{1}{h^2} \left(-j\omega\varepsilon \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial y} \right)$$

For different modes
TEM : $E_z = H_z = 0$
TE : $E_z = 0$
TM : $H_z = 0$

For TEM mode to be exist, $h = 0$ so that $\gamma = \sqrt{h^2 - k^2} = jk = \alpha + j\beta$ that $\beta = k$

For TE, TM mode, $\beta \neq k$, so

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{-k^2 \left(1 - \frac{h^2}{k^2} \right)} = jk \sqrt{1 - \left(\frac{h}{k} \right)^2}$$

Where

$$\frac{h}{k} = \frac{h}{\omega\sqrt{\mu\varepsilon}} = \frac{h}{2\pi f \sqrt{\mu\varepsilon}} = \frac{f_c}{f} \qquad f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

Thus

$$\gamma = jk \sqrt{1 - \left(\frac{f_c}{f} \right)^2}$$

Evanescent mode and Propagation mode

$$\gamma = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

When $f_c < f$

$$\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \pm j\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

Select $-j$

$$\gamma = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2} = k\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} = \alpha \in \mathbb{R}$$

Then

$$e^{-\gamma z} = e^{-\alpha z} \quad \text{Decaying Evanescent mode}$$

When $f_c > f$

$$\gamma = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2} = j\beta \in \text{Pure imaginary}$$

$$e^{-\gamma z} = e^{-\beta z} \quad \text{Propagating mode}$$

TE, TM modes

The general waves

$$\widetilde{E}_X = \frac{1}{h^2}\left(-j\omega\mu\frac{\partial H_z}{\partial y} - \gamma\frac{\partial E_z}{\partial x}\right) \qquad \widetilde{H}_X = \frac{1}{h^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right)$$

$$\widetilde{E}_Y = \frac{1}{h^2}\left(j\omega\mu\frac{\partial H_z}{\partial y} - \gamma\frac{\partial E_z}{\partial y}\right) \qquad \widetilde{H}_Y = \frac{1}{h^2}\left(-j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial y}\right)$$

Consider ideal waveguide (no waveguide loss) $\gamma = j\beta$, $\alpha = 0$

$$\widetilde{E}_X = \frac{1}{h^2}\left(-j\omega\mu\frac{\partial H_z}{\partial y} - j\beta\frac{\partial E_z}{\partial x}\right) \qquad \widetilde{H}_X = \frac{1}{h^2}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y} - j\beta\frac{\partial H_z}{\partial x}\right)$$

$$\widetilde{E}_Y = \frac{1}{h^2}\left(j\omega\mu\frac{\partial H_z}{\partial y} - j\beta\frac{\partial E_z}{\partial y}\right) \qquad \widetilde{H}_Y = \frac{1}{h^2}\left(-j\omega\varepsilon\frac{\partial E_z}{\partial y} - j\beta\frac{\partial H_z}{\partial y}\right)$$

When TE mode, $E_z = 0$

$$\widetilde{E}_X = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \qquad \widetilde{H}_X = \frac{-j\beta}{h^2}\frac{\partial H_z}{\partial x}$$

$$\widetilde{E}_Y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \qquad \widetilde{H}_Y = \frac{1-j\beta}{h^2}\frac{\partial H_z}{\partial y}$$

When TM mode, $H_z = 0$

$$\widetilde{E}_X = \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial x} \qquad \widetilde{H}_X = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$\widetilde{E}_Y = \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial y} \qquad \widetilde{H}_Y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

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