Parallel Plate Waveguide

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Imagine two parallel plates. Along x-axis, the plates have width w. Along y-axis, two plate separated by a gap d. (Thus, the lower plate is placed at $y = 0$ and the upper plate is placed at $y = d$). And, $w \gg d$.

By laws of electromagnetics, we have

$$\nabla^2 \Phi(x, y) = 0 \qquad \begin{cases} 0 \leq x \leq W \\ 0 \leq y \leq d \end{cases}$$

$$\text{Boundary Conditions} \begin{cases} \Phi(x, 0) = 0 \\ \Phi(x, d) = V_0 \end{cases}$$

Since $W \gg d$, thus the wave should not have variation along x : $\dfrac{\partial}{\partial x} = 0$. That implies Φ is indepdent of x, $\Phi(x, y) = \Phi(y)$ and thus

$$\nabla^2 \Phi(y) = 0 \qquad \begin{cases} 0 \leq x \leq W \\ 0 \leq y \leq d \end{cases}$$

Then

$$\nabla^2 \Phi(y) = 0 \qquad \iff \qquad \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{0} + \frac{\partial^2}{\partial y^2} \right) \Phi(y) = 0 \qquad \iff \frac{\partial^2}{\partial y^2} \Phi(y) = \frac{d^2 \Phi(y)}{dy^2} = 0$$

i.e.

$$\frac{d}{dy} \frac{d\Phi(y)}{dy} = 0 \iff \frac{d\Phi(y)}{dy} = B \iff \Phi(y) = A + By$$

Using boundary conditions

$$\text{Boundary Conditions} \begin{cases} \Phi(0) = 0 \\ \Phi(d) = V_0 \end{cases}$$

$$\Phi(0) = A = 0 \qquad \text{and} \qquad \Phi(d) = Bd = V_0 \iff B = \frac{V_0}{d}$$

Therefore the potential is

$$\Phi(y) = \frac{V_0}{d} y$$

Since wave propagate along z direction to the infinity, so the propagation term has $e^{-j\beta z}$

Therefore the E, H can be found :

$$E(x,y,z) = [-\nabla\Phi(x,y)]\, e^{-j\beta z} = \left[-\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\frac{V_0}{d}y\right] e^{-j\beta z} = -\frac{V_0}{d}e^{-j\beta z}\hat{y}$$

$$H(x,y,z) = \frac{\hat{z}}{\eta} \times E = -\frac{V_0}{\eta d}e^{j\beta z}\,(\hat{z}\times\hat{y}) = -\frac{V_0}{\eta d}e^{j\beta z}(-\hat{x}) = \frac{V_0}{\eta d}e^{j\beta z}\hat{x}$$

Therefore the voltage and current

$$EMF = V = -\int_{y=0}^{y=d} E \cdot d\bar{y} = \int_0^d -\frac{V_0}{d}e^{-j\beta z}\hat{y}\cdot\hat{y}dy = -\frac{V_0}{d}e^{-j\beta z}\int_0^d dy = V_0 e^{-j\beta z}$$

$$MMF = I = \oint \bar{J}_S \cdot (\hat{z}dx)\, d\bar{x} = \int_{x=0}^{w} (-\hat{y}\times H)\cdot(\hat{z}dx) = \int_{x=0}^{w}\frac{V_0}{\eta d}e^{j\beta z}dx\,\underbrace{(-\hat{y}\times\hat{x})\cdot\hat{z}}_{1} = \frac{V_0}{\eta}\frac{w}{d}e^{j\beta z}$$

Impedance of the waveguide

$$Z = \frac{V}{I} = \frac{V_0 e^{-j\beta z}}{\frac{V_0}{\eta}\frac{w}{d}e^{j\beta z}} = \frac{d}{w}\eta = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$$

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