Dielectric Slab Waveguide (Step-index optical fiber)

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1. Derive the 6 E, H relations from Maxwell's Equations

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x(x,y,z) & E_y(x,y,z) & E_z(x,y,z) \end{vmatrix} = -\mu \left(\frac{\partial H_x}{\partial t}, \frac{\partial H_y}{\partial t}, \frac{\partial H_z}{\partial t} \right)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x(x,y,z) & H_y(x,y,z) & H_z(x,y,z) \end{vmatrix} = \varepsilon \left(\frac{\partial E_x}{\partial t}, \frac{\partial E_y}{\partial t}, \frac{\partial E_z}{\partial t} \right)$$

Compare the component , i use $[\,]_A$ to denote $\dfrac{\partial}{\partial A}$

$$\begin{array}{ll} [E_z]_y - [E_y]_z = -\mu [H_x]_t & [H_z]_y - [H_y]_z = \varepsilon [E_x]_t \\ [E_x]_z - [E_z]_x = -\mu [H_y]_t & [H_x]_z - [H_z]_x = \varepsilon [E_y]_t \\ [E_y]_x - [E_x]_y = -\mu [H_z]_t & [H_y]_x - [H_x]_y = \varepsilon [E_z]_t \end{array}$$

2. Case analysis for field simplifications

Now consider the following conditions:

1. Assume fields are time harmonic : $E(x,y,z,t)$ can be written as phasor that $E(x,y,z,t) = Re\left[E(x,y,z)e^{j\omega t}\right]$. then $\dfrac{\partial}{\partial t} = j\omega$ in phasor form

2. Assume propagation along z , then the propagation term $e^{-j\beta z}$ can be extracted out as $E(x,y,z) = E(x,y)e^{-j\beta z}$ (in phasor) , that means the amplitude term is independent of z and it implies $\dfrac{\partial}{\partial z} = j\beta$.

3. Assumes $y \to \infty$ (infinite long slab plate along y-direction), so the physical condition has no vaiation along y-axis and thus there should be no variation in y : $E(x,y,z,t) = E(x,z,t)$, then $\dfrac{\partial}{\partial y} = 0$.

4. Assume it is TE wave, then $E_z = 0$, $H_z \neq 0$ (if $H_z = 0$ then it become TEM wave , not TE wave).

3. Update the 6 E, H relations

Apply condition $\dfrac{\partial}{\partial y} = 0$

$$\begin{array}{ll} \underbrace{[E_z]_y}_{0} - [E_y]_z = -\mu [H_x]_t & \underbrace{[H_z]_y}_{0} - [H_y]_z = \varepsilon [E_x]_t \\ [E_x]_z - [E_z]_x = -\mu [H_y]_t & [H_x]_z - [H_z]_x = \varepsilon [E_y]_t \\ [E_y]_x - \underbrace{[E_x]_y}_{0} = -\mu [H_z]_t & [H_y]_x - \underbrace{[H_x]_y}_{0} = \varepsilon [E_z]_t \end{array}$$

Then apply the condition $E_z = 0$

$$[E_y]_z = \mu [H_x]_t \qquad -[H_y]_z = \varepsilon [E_x]_t$$

$$[E_x]_z - \left[\underbrace{E_z}_{0}\right]_x = -\mu [H_y]_t \qquad [H_x]_z - [H_z]_x = \varepsilon [E_y]_t$$

$$[E_y]_x = -\mu [H_z]_t \qquad [H_y]_x = \varepsilon \left[\underbrace{E_z}_{0}\right]_t$$

Then H_y is zero and thus E_x is also zero

$$[E_y]_z = \mu [H_x]_t \qquad \left\{-[H_y]_z = \varepsilon [E_x]_t\right\}$$

$$\left\{[E_x]_z = -\mu [H_y]_t\right\} \qquad [H_x]_z - [H_z]_x = \varepsilon [E_y]_t$$

$$[E_y]_x = -\mu [H_z]_t \qquad \left\{[H_y]_x = 0\right\}$$

Finally we have 3 equations left

$$[E_y]_z = \mu [H_x]_t \qquad [E_y]_x = -\mu [H_z]_t \qquad [H_x]_z - [H_z]_x = \varepsilon [E_y]_t$$

Apply $\dfrac{\partial}{\partial t} = j\omega$

$$[E_y]_z = -j\omega\mu H_x \qquad [E_y]_x = -j\omega\mu H_z \qquad [H_x]_z - [H_z]_x = j\omega\varepsilon E_y$$

Apply $\dfrac{\partial}{\partial z} = j\beta$

$$j\beta E_y = -j\omega\mu H_x \qquad \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \qquad j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

4. Solve the wave equations as a ODE problem

$$\nabla^2 \left\{ \begin{array}{c} E \\ H \end{array} \right\} + \omega^2\mu\varepsilon \left\{ \begin{array}{c} E \\ H \end{array} \right\} = 0$$

Put $E(x,y,z) = E_y(x)e^{-j\beta z}$ into the wave eqautions

$$\nabla^2 \left\{E_y(x)e^{-j\beta z}\right\} + \omega^2\mu\varepsilon \left\{E_y(x)e^{-j\beta z}\right\} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \underbrace{\frac{\partial^2}{\partial y^2}}_{0} + \frac{\partial^2}{\partial z^2}\right)\left\{E_y(x)e^{-j\beta z}\right\} + \omega^2\mu\varepsilon E_y(x)e^{-j\beta z} = 0$$

$$\left(e^{-j\beta z}\frac{\partial^2 E_y(x)}{\partial x^2} + E_y(x)\frac{\partial^2 e^{-j\beta z}}{\partial z^2}\right) + \omega^2\mu\varepsilon E_y(x)e^{-j\beta z} = 0$$

$$e^{-j\beta z}\frac{\partial^2 E_y(x)}{\partial x^2} - \beta^2 E_y(x)e^{-j\beta z} + \omega^2\mu\varepsilon E_y(x)e^{-j\beta z} = 0$$

$$\frac{\partial^2 E_y(x)}{\partial x^2} + \left(\omega^2\mu\varepsilon - \beta^2\right) E_y(x) = 0$$

Let $\omega^2\mu\varepsilon - \beta^2 = k^2_{cutoff}$, then the problem now becomes a ODE

$$\frac{\partial^2 E_y(x)}{\partial x^2} + k^2_{cutoff} E_y(x) = 0$$

To solve this 2nd order ODE, let $E_y(x) = Ae^{px}$, plug it into the equation

$$Ap^2 e^{px} + k_{cutoff}^2 A e^{px} = 0$$

$$\iff \quad p^2 + k_{cutoff}^2 = 0$$

$$\iff \quad p = \pm j k_{cutoff}$$

Thus the general solution of this equation (inside core layer), which is the superposition of all possible solutions (actually only 2), is

$$A \sin k_c x + B \cos k_c x \qquad\qquad |x| \leq d$$

where d is the slap thickness.

The unknown A, B can be solved by using boundary condition.

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