Scalar and Vector Potentials, Gauge Conditions

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1 The vector potential A

Consider EM fields in free space

$$\begin{cases} \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \times H = \dfrac{\partial D}{\partial t} + J_c \\ \nabla \cdot D = \rho \\ \nabla \cdot B = 0 \end{cases}$$

Since B field is non-divergent ($\nabla \cdot B = 0$), so we can define

$$B = \nabla \times A$$

What A means : The close loop line integral of A = flux pass through that loop.

Unlike electrostatics that E is created by charges, in electrodynamics E is created by charges and induced by B, thus a single potential is not sufficient to describe E

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t} \\ &= -\nabla \times \frac{\partial A}{\partial t} \end{aligned}$$

$$\iff \quad \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

$$\iff E + \frac{\partial A}{\partial t} \text{ is curl-less}$$

$$\iff E + \frac{\partial A}{\partial t} = -\nabla \phi$$

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

Thus

$$B = \nabla \times A \qquad E = -\nabla \phi - \frac{\partial A}{\partial t}$$

Vector potential A and scalar potential ϕ now descripe B and E.

* The ϕ now is not the electric potential. Thus when frequency is high, the electric potential in electrostatic case is not valid anymore.

2 Gauge

It is necessary to force some constraints on the definition of A such that the potentials can *uniquely* describe E and B. Without such constraint, the description is not unique.

For example, consider potentials A and ϕ

$$B = \nabla \times A \qquad E = -\nabla\phi - \frac{\partial A}{\partial t}$$

Now consider another paris of potentials A' and ϕ'

$$A' = A + \nabla\psi$$

$$\phi' = \phi - \frac{\partial \psi}{\partial t}$$

Then because curl grad $= 0$

$$\nabla \times A' = \nabla \times A + \underbrace{\nabla \times \nabla\psi}_{0} = B$$

And because of mixed derivative equality $\dfrac{\partial f}{\partial x_1 \partial x_2} = \dfrac{\partial f}{\partial x_2 \partial x_1}$

$$\begin{aligned} -\nabla\phi' - \frac{\partial A'}{\partial t} &= \left(-\nabla\phi + \nabla\frac{\partial\psi}{\partial t}\right) - \left(\frac{\partial A}{\partial t} + \frac{\partial}{\partial t}\nabla\psi\right) \\ &= \left(-\nabla\phi - \frac{\partial A}{\partial t}\right) + \left(\underbrace{\nabla\frac{\partial\psi}{\partial t} - \frac{\partial}{\partial t}\nabla\psi}_{0}\right) \\ &= E \end{aligned}$$

Thus two potentials are describing the same thing and there is no way to differentiate them.

From observation, such non-uniqueness exists because there is no limitation on the divergence of A. Since B and E do not enforce any limitation on $\nabla \cdot A$, thus we have many different ways to define $\nabla \cdot A$. Different limitation on $\nabla \cdot A$ fits different problem. Thus , a *Gauge* , **is a imposed limitations that cope with the redundant extra degrees of freedom.** There are 2 famous gauge in electromagnetics, the Coulomb gauge $\nabla \cdot A = 0$ and the Lorenz gauge $\nabla \cdot A + \dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t} = 0$.

3 d'Alembert equations and the Gauges

With these equations

$$\begin{cases} \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \times H = \dfrac{\partial D}{\partial t} + J_c \\ \nabla \cdot D = \rho \\ \nabla \cdot B = 0 \end{cases} \qquad \begin{cases} B = \nabla \times A \\ E = -\nabla\phi - \dfrac{\partial A}{\partial t} \end{cases}$$

The Ampere's Law thus becomes

$$
\begin{aligned}
\nabla \times H &= \frac{\partial D}{\partial t} + J_c \\
\nabla \times B &= \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} + \mu_0 J_c \\
\nabla \times \nabla \times A &= \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left[-\nabla\phi - \frac{\partial A}{\partial t}\right] + \mu_0 J_c \\
\nabla(\nabla \cdot A) - \nabla^2 A &= -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} + \mu_0 J_c
\end{aligned}
$$

Rearrange

$$
\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}\right) = -\mu_0 J
$$

Next, consider the Gauss's Law for electric field

$$
\begin{aligned}
\nabla \cdot D &= \rho \\
\nabla \cdot E &= \frac{\rho}{\varepsilon} \\
\nabla \cdot \left(-\nabla\phi - \frac{\partial A}{\partial t}\right) &= \frac{\rho}{\varepsilon} \\
-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot A &= \frac{\rho}{\varepsilon}
\end{aligned}
$$

Thus we have the following 2 d'alembert equations

$$
\begin{cases}
\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}\right) = -\mu_0 J \\
-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot A = \frac{\rho}{\varepsilon}
\end{cases}
$$

Just by looking at the second equation, if $\nabla \cdot A = 0$ (Coulomb gauge) :

$$
\begin{cases}
\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi = -\mu_0 J \\
\nabla^2\phi = -\frac{\rho}{\varepsilon}
\end{cases}
$$

The second equation is the same Laplace equation in electrostatics.

If Lorenz gauge is used $\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t} = 0$, the d'Alembert equations become a symmetric form :

$$
\begin{cases}
\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 J & \text{Current generated} \\
\nabla^2\phi - \mu_0 \varepsilon_0 \frac{\partial^2 \phi}{\partial t} = -\frac{\rho}{\varepsilon} & \text{Charge generated}
\end{cases}
$$

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