Dipole Antenna / Hertzian dipole

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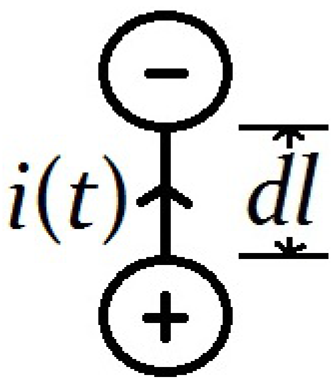
1. The **A**
2. The **B** and **H**
3. The **E**
4. Far field approximation
5. The **S** , P_{rad} and R_{rad}

- The Hertzian dipole does not exists in real life, but the real antenna can be formed by taking integration on Hertzian dipole.



Current element / dipole carry dynamic current

$dl \leq \dfrac{\lambda}{10}$, it is very small

$i(t) = I_0 \cos \omega t$, the current on the dipole

It is difficult to solve for the **B** and **E** directly.

Use Retarded Potential : $V(x, y, z, t) \longrightarrow V\left(x, y, z, t - \frac{r}{c}\right)$, so the retarded current is

$$i(t) = I_0 \cos \omega \left(t - \tfrac{r}{c}\right) = I_0 \cos(\omega t - \tfrac{\omega}{c} r) = I_0 \cos(\omega t - \beta r)$$

Retarded : time delay by $\frac{r}{c}$ or phase delay by βr

1 The retarded magnetic vector potential

$$\mathbf{A}(r,t) = \frac{\mu[i(t)]dl}{4\pi r}\hat{z} = \frac{\mu I_0 \cos\left(t - \frac{r}{c}\right) dl}{4\pi r}\hat{z}$$

In phasor form

$$\widetilde{\mathbf{A}}(r) = \frac{\mu I_0 dl}{4\pi r} e^{-j\beta r}\,\hat{z}$$

(Recall) Phasor transformation : $V_0 \cos(\omega t + \phi) = \Re e\left(V_0 e^{j(\omega t+\phi)}\right) = \Re e\left(\underbrace{V_0 e^{j\phi}}_{Phasor} e^{j\omega t}\right)$

For radiation by antenna, it is more convenient to select spherical coordinate system

(Recall) The coordinate transform of rectangular to spherical $V(x,y,z) \to V(r,\theta,\phi)$
Given vector $\bar{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$, the vector representation in spherical coordinate is

$$\hat{V} = (V_x \sin\theta\cos\phi + V_y \sin\theta\sin\phi + V_z\cos\theta)\,\hat{r} + (-V_x\sin\phi + V_y\cos\phi)\,\hat{\phi} + (V_x\cos\theta\cos\phi + V_y\cos\theta\sin\phi - V_z\sin\theta)\,\hat{\theta}$$

Since $\widetilde{\mathbf{A}}(r) = \frac{\mu I_0}{4\pi r} dl\, e^{-j\beta r}\,\hat{z} = A_z\hat{z}$ only, $A_x = A_y = 0$, thus

$$\widetilde{\mathbf{A}} = (A_z\cos\theta)\,\hat{r} + (-A_z\sin\theta)\,\hat{\theta}$$

$$= \frac{\mu I_0 dl}{4\pi r}\cos\theta e^{-j\beta r}\hat{r} - \frac{\mu I_0 dl}{4\pi r}\sin\theta e^{-j\beta r}\hat{\theta}$$

$$= \frac{\mu I_0 dl}{4\pi}\left(\frac{\cos\theta}{r}e^{-j\beta r}\hat{r} - \frac{\sin\theta}{r}e^{-j\beta r}\hat{\theta}\right)$$

- $\mathbf{A}(r,\theta)$ is ϕ independent $\iff \frac{\partial}{\partial\phi} = 0$
- $\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} \iff A_\phi = 0$

2 B and H

Find $\mathbf{B}$ by $\mathbf{B} = \nabla \times \mathbf{A}$

$$\widetilde{\mathbf{B}} = \nabla \times \widetilde{\mathbf{A}} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Since $\mathbf{A}$ is ϕ independent $\Rightarrow \frac{\partial}{\partial\phi} = 0$, and $A_\phi = 0$

$$\widetilde{\mathbf{B}} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0 \\ A_r & rA_\theta & 0 \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} \\ A_r & rA_\theta \end{vmatrix}\hat{\phi} = \frac{1}{r}\left(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

$$= \frac{1}{r}\frac{\mu I_0 dl}{4\pi}\left(-\frac{\partial}{\partial r}\sin\theta e^{-j\beta r} - \frac{\partial}{\partial\theta}\frac{\cos\theta}{r}e^{-j\beta r}\right)\hat{\phi}$$

$$= \frac{\mu I_0 dl}{4\pi r}\left(\sin\theta j\beta + \frac{e^{-j\beta r}}{r}\sin\theta\right)\hat{\phi} = \frac{\mu I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\hat{\phi}$$

Thus, by $\mathbf{H} = \frac{\mathbf{B}}{\mu}$

$$\widetilde{\mathbf{H}} = \frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\hat{\phi}$$

- $H_r = H_\theta = 0$
- $\mathbf{H}(r,\theta)$ is ϕ independent $\iff \frac{\partial}{\partial\phi} = 0$

3 The E

Find $\mathbf{E}$ by $\varepsilon\frac{\partial\mathbf{E}}{\partial t} = \nabla\times\mathbf{H}$

In phasor form, $\varepsilon j\omega\widetilde{\mathbf{E}} = \nabla\times\widetilde{\mathbf{H}}$, thus

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

Since $\mathbf{H}$ is ϕ independent , $\frac{\partial}{\partial\phi} = 0$, and $H_r = H_\theta = 0$

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & r\sin\theta H_\phi \end{vmatrix} = \frac{1}{j\omega\varepsilon}\left(\frac{\hat{r}}{r^2\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial\theta} - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial r}\right)$$

$H_\phi = \frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)$, so $r\sin\theta H_\phi = \frac{I_0 dl}{4\pi}\sin^2\theta e^{-j\beta r}\left(j\beta + \frac{1}{r}\right)$
and thus

$$\frac{\partial r\sin\theta H_\phi}{\partial\theta} = \frac{I_0 dl}{4\pi}2\sin\theta\cos\theta e^{-j\beta r}\left(j\beta + \frac{1}{r}\right)$$

$$\frac{\partial r\sin\theta H_\phi}{\partial r} = \frac{I_0 dl}{4\pi}\sin^2\theta\left[-j\beta e^{-j\beta r}\left(j\beta + \frac{1}{r}\right) + e^{-j\beta r}\left(\frac{-1}{r^2}\right)\right] = \frac{I_0 dl}{4\pi}\sin^2\theta\left[\beta^2 e^{-j\beta r} - \frac{j\beta e^{-j\beta}}{r} - \frac{e^{-j\beta r}}{r^2}\right]$$

And therefore

$$\frac{\hat{r}}{r^2\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial\theta} = \frac{I_0 dl}{2\pi}\cos\theta e^{-j\beta r}\left(\frac{j\beta}{r^2} + \frac{1}{r^3}\right)\hat{r}$$

$$\frac{\hat{\theta}}{r\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial r} = \frac{I_0 dl}{4\pi}\sin\theta\left[\frac{\beta^2 e^{-j\beta r}}{r} - \frac{j\beta e^{-j\beta r}}{r^2} - \frac{e^{-j\beta r}}{r^3}\right]\hat{\theta}$$

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\left(\frac{\hat{r}}{r^2\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial\theta} - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial r\sin\theta H_\phi}{\partial r}\right)$$

$$= \frac{1}{\omega\varepsilon}\frac{I_0 dl}{2\pi}\cos\theta e^{-j\beta r}\left(\frac{\beta}{r^2} - \frac{j}{r^3}\right)\hat{r} - \frac{1}{\omega\varepsilon}\frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left[\frac{\beta^2}{jr} - \frac{\beta}{r^2} - \frac{1}{jr^3}\right]\hat{\theta}$$

Since $\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\beta}{\omega\epsilon}$

$$\widetilde{\mathbf{E}} = \eta\frac{I_0 dl}{2\pi}\cos\theta e^{-j\beta r}\left(\frac{1}{r^2} - \frac{j}{\beta r^3}\right)\hat{r} + \eta\frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left[\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right]\hat{\theta}$$

- $E_\phi = 0$
- E is ϕ independent $\iff \frac{\partial}{\partial\phi} = 0$

4 Far Field Approximations

Therefore the fields are

$$\widetilde{\mathbf{H}} = \frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\hat{\phi}$$

$$\widetilde{\mathbf{E}} = \eta\frac{I_0 dl}{2\pi}\cos\theta e^{-j\beta r}\left(\frac{1}{r^2} - \frac{j}{\beta r^3}\right)\hat{r} + \eta\frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\left[\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right]\hat{\theta}$$

When r is large (far field) , ignore all the term with power ≥ 2

$$\widetilde{\mathbf{H}}(r,\theta) = \frac{I_0 dl}{4\pi}\sin\theta e^{-j\beta r}\frac{j\beta}{r}\hat{\phi} \qquad \widetilde{\mathbf{E}}(r,\theta) = \eta\frac{I_0 dl}{4\pi}\sin\theta\frac{j\beta}{r}\hat{\theta}$$

In time domain

$$\mathbf{H}(r,\theta,t) = -\frac{I_0\beta dl}{4\pi r}\sin\theta\sin(\omega t - \beta r)\hat{\phi} \qquad \widetilde{\mathbf{E}} = -\eta\frac{I_0\beta dl}{4\pi r}\sin\theta\sin(\omega t - \beta r)\hat{\theta}$$

5 Poynting Vector S , $\mathbf{S}_{avg}$, P_{rad} and R_{rad}

The Poynting vector **S** in time domain is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \eta\frac{I_0^2\beta^2 dl^2}{16\pi^2 r^2}\sin^2\theta\sin^2(\omega t - \beta r)\hat{r}$$

The time-average Poynting vector in phasor domain is

$$\widetilde{\mathbf{S}}_{\mathbf{avg}} = \frac{1}{2}\Re e\left[\widetilde{\mathbf{E}}\times\widetilde{\mathbf{H}}^*\right] = \frac{1}{2}\Re e\left[E_\theta H_\phi^*\right]\hat{r} = \frac{1}{2}\eta|H_\phi|^2\hat{r} = \eta\frac{I_0^2\beta^2 dl^2}{32\pi^2 r^2}\sin^2\theta\hat{r}$$

Thus the time-average radiated power is

$$P_{rad} = \oiint_S \widetilde{\mathbf{S}}_{avg} \cdot d\bar{\mathbf{S}} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\frac{\eta I_0^2 \beta^2 dl^2}{32\pi^2 r^2} \sin^2\theta \right) \left(r^2 \sin\theta d\theta d\phi \right)$$

$$= \frac{\eta I_0^2 \beta^2 dl^2}{32\pi^2} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin^3\theta d\theta$$

- $\beta^2 = \left(\frac{2\pi}{\lambda} \right)^2 = \frac{4\pi^2}{\lambda^2}$
- $\int_{\phi=0}^{2\pi} d\phi = 2\pi$
- $\int_{\theta=0}^{\pi} \sin^3\theta d\theta = -\int_{\theta=0}^{\pi} (1 - \cos^2\theta)\, d\cos\theta = -\left[\cos\theta - \frac{\cos^3\theta}{3} \right]_0^{\pi} = -\left[-2 - \frac{-2}{3} \right] = \frac{4}{3}$

$$P_{rad} = \frac{\eta I_0^2 \left(\frac{4\pi^2}{\lambda^2} \right) dl^2}{32\pi^2} 2\pi \cdot \frac{4}{3} = \frac{\eta I_0^2 \pi}{3} \left(\frac{dl}{\lambda} \right)^2$$

Special case, in free space, $\eta_0 \approx 120\pi$

$$P_{rad} = I_0^2 40\pi^2 \left(\frac{dl}{\lambda} \right)^2$$

Since $P = I^2 R$ (for static) $P = \frac{1}{2} I_0^2 R$ (for dynamic), thus the fictitious resistance, the radiation resistance R_{rad} (in free space) is

$$R_{rad} = 80\pi^2 \left(\frac{dl}{\lambda} \right)^2$$

Where the general radiation resistance

$$R_{rad} = \frac{2\eta\pi}{3} \left(\frac{dl}{\lambda} \right)^2$$