The Array Factor of simple linear uniform antenna array

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Antenna array can be treated as ONE composite antenna

For uniform linear array with : same type
same orientation
same separation
same amplitude of current
phase delay as a A.S.

Consider the far field ($d \gg l$), then resultant field of the array can be treated as a scalar sum of the field of each antenna element.

The radiation far field of the i^{th} antenna element is

$$E_i = \frac{C_i I_i}{r_i} f_i(\theta, \phi) e^{-j(kr_i + \psi_i)}$$

same type	$C_1 = C_2 = ... = C_n = C$
same orientation	$f_1 = f_2 = ... = f_n = f$
for far field	$\frac{1}{r_1} = \frac{1}{r_2} = ... = \frac{1}{r_n} = \frac{1}{r}$
same amplitude of current	$I_1 = I_2 = ... = I_n = I$

So

$$E_i = \frac{CI}{r} f(\theta, \phi) e^{-j(kr_i + \psi_i)}$$

For arithmetic sequence phase delay

$$\begin{gathered} \psi_1 = \psi_1 \\ \psi_2 = \psi_1 + \Delta\psi \\ \psi_3 = \psi_1 + 2\Delta\psi \\ \vdots \\ \psi_n = \psi_1 + (n-1)\Delta\psi \end{gathered}$$

For same separation

$$\begin{gathered} e^{-jkr_1} = \exp(-jkr_1) \\ e^{-jkr_2} = \exp(-jk[r_1 - d\cos\theta]) = \exp(-jkr_1)\exp(jkd\cos\theta) \\ e^{-jkr_3} = \exp(-jk[r_1 - 2d\cos\theta]) = \exp(-jkr_1)\exp(jk2d\cos\theta) \\ \vdots \\ e^{-jkr_n} = \exp(-jk[r_1 - (n-1)d\cos\theta]) = \exp(-jkr_1)\exp(jk(n-1)d\cos\theta) \end{gathered}$$

Thus

$$E_1 = \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(-j\psi_1)$$

$$E_2 = \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jkd\cos\theta) \exp(-j\psi_1) \exp(-j\Delta\psi)$$

$$E_3 = \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jk2d\cos\theta) \exp(-j\psi_1) \exp(-j2\Delta\psi)$$

$$\vdots$$

$$E_n = \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jk(n-1)d\cos\theta) \exp(-j\psi_1) \exp(-j(n-1)\Delta\psi)$$

Since resultant far field of the array can be treated as a scalar sum of the field of each antenna element, thus

$$E_{Resultant} = \sum E_i = E_1 + E_2 + ... + E_n$$

$$\begin{aligned} = & \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(-j\psi_1) \\ & + \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jkd\cos\theta) \exp(-j\psi_1) \exp(-j\Delta\psi) \\ & + \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jk2d\cos\theta) \exp(-j\psi_1) \exp(-j2\Delta\psi) \\ & + \quad ... \\ & + \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(jk(n-1)d\cos\theta) \exp(-j\psi_1) \exp(-j(n-1)\Delta\psi) \end{aligned}$$

$$= \frac{CI}{r} f(\theta, \phi) \exp(-jkr_1) \exp(-j\psi_1) \begin{pmatrix} & 1 \\ + & \exp(jkd\cos\theta) \exp(-j\Delta\psi) \\ + & \exp(jk2d\cos\theta) \exp(-j2\Delta\psi) \\ + & ... \\ + & \exp(jk(n-1)d\cos\theta) \exp(-j(n-1)\Delta\psi) \end{pmatrix}$$

$$= \frac{CI}{r} f(\theta, \phi) \exp[-j(kr_1 + \psi_1)] \begin{pmatrix} & \exp[j \cdot 0 \cdot (kd\cos\theta - \Delta\psi)] \\ + & \exp[j \cdot 1 \cdot (kd\cos\theta - \Delta\psi)] \\ + & \exp[j \cdot 2 \cdot (kd\cos\theta - \Delta\psi)] \\ + & ... \\ + & \exp[j \cdot (n-1) \cdot (kd\cos\theta - \Delta\psi)] \end{pmatrix}$$

$$\textbf{Handling}\begin{pmatrix} & \exp\left[j\cdot 0\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & \exp\left[j\cdot 1\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & \exp\left[j\cdot 2\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & ... \\ + & \exp\left[j\cdot(n-1)\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \end{pmatrix}$$

Using the techniques in deriving Dirichlet Function

Consider following sum (which is a geomeric series)

$$1+e^{j\theta}+e^{j2\theta}+...+e^{j(n-3)\theta}+e^{j(n-2)\theta}+e^{j(n-1)\theta}$$

By $1+r+r^2+...+r^{n-1}=\dfrac{1-r^N}{1-r}\ (|r|<1)$

$$=\frac{1-e^{jn\theta}}{1-e^{j\theta}}$$

Then apply a trick

$$1-e^{jn\theta}=e^{jn\theta/2}\left(e^{-jn\theta/2}-e^{jn\theta/2}\right) \qquad 1-e^{j\theta/2}=e^{j\theta/2}\left(e^{-j\theta/2}-e^{j\theta/2}\right)$$

$$\frac{1-e^{jn\theta}}{1-e^{j\theta}}=\frac{e^{jn\theta/2}\left(e^{-jn\theta/2}-e^{jn\theta/2}\right)}{e^{j\theta/2}\left(e^{-j\theta/2}-e^{j\theta/2}\right)}$$

$$=e^{j(n-1)\theta/2}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

Thus

$$1+e^{j\theta}+e^{j2\theta}+...+e^{j(n-3)\theta}+e^{j(n-2)\theta}+e^{j(n-1)\theta}=e^{j(n-1)\theta/2}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

Apply this to

$$\begin{pmatrix} & \exp\left[j\cdot 0\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & \exp\left[j\cdot 1\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & \exp\left[j\cdot 2\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \\ + & ... \\ + & \exp\left[j\cdot(n-1)\cdot\left(kd\cos\theta-\Delta\psi\right)\right] \end{pmatrix}=\exp\left[j\cdot\frac{n-1}{2}\cdot\left(kd\cos\theta-\Delta\psi\right)\right]\frac{\sin\left[\frac{n}{2}\left(kd\cos\theta-\Delta\psi\right)\right]}{\sin\left[\frac{1}{2}\left(kd\cos\theta-\Delta\psi\right)\right]}$$

Thus the resultant far field of the antenna array is now equal to

$$E_{Resultant}=\frac{CI}{r}f\left(\theta,\phi\right)\exp\left[-j\left(kr_1+\phi_1\right)\right]\underbrace{\left(\exp\left[j\cdot\frac{n-1}{2}\cdot\left(kd\cos\theta-\Delta\psi\right)\right]\frac{\sin\left[\frac{n}{2}\left(kd\cos\theta-\Delta\psi\right)\right]}{\sin\left[\frac{1}{2}\left(kd\cos\theta-\Delta\psi\right)\right]}\right)}_{array\,factor\ ,\ f_n}$$

The function f_n looks like the Dirichlet Function in DSP.

Thus the resultant E field (far field) is thus

$$E_{Resultant}=\frac{CI}{r}e^{-j(kr_1+\psi_1)}f\left(\theta,\phi\right)f_n\left(k,d,\theta,\psi\right)$$

$-END-$