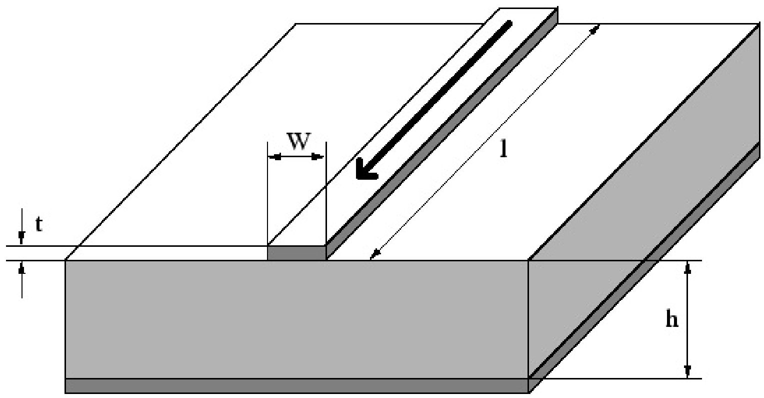
Lumped Circuit Model of Transmission Line and Telegrapher's Equation

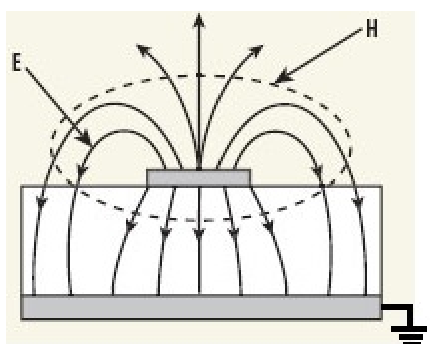
September 10, 2013

1 The Structure of a microstrip line

The following diagram is *microstrip line* , which is the most simple form of transmission line



The general structure of a microstrip line cosist of a ground plate ($V = 0$) with a thin metal plate with high potential on the top. Dielectric ε_r is filled in between the two line. Following is the front view of the line, the diagram shows the field distribution.



Signals (which can be current or voltage) will *flow* / *propagate* along the transmission line.

2 Descriptions of the characteristics of the transmission line and the circuit model

The Shunt Capacitance

Since there are 2 metal plates, one with high potential and one with zero peotential , so there is a shunt capacitance there

The Series Resistance

Since there is ohmic loss in the line , thus there is a serial resistance in there

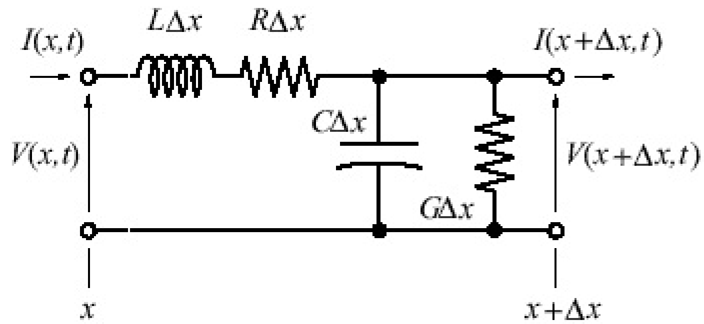
The Shunt Conductance

Since there is dielectric loss between 2 plate, thus there is a shunt conductance there

The Series Inductance

Since current will change along the transimission line, thus there is a series inductance there

The complete Circuit Model of Transmission Line



Notice that the unit of all the circuit element are per unit length

3 Telegrapher's Equation

Apply KVL on the transmission line

$$V(x,t) - L\Delta x \frac{\partial I(x,t)}{\partial t} - R\Delta x I(x,t) = V(x+\Delta x, t)$$

Apply KCL on the transmission line

$$I(x,t) - C\Delta x \frac{\partial V(x,t)}{\partial t} - G\Delta x V(x,t) = I(x+\Delta x, t)$$

Thus the 2 equations are

$$\begin{cases} V(x,t) - L\Delta x \frac{\partial I(x,t)}{\partial t} - R\Delta x I(x,t) = V(x+\Delta x, t) \\ I(x,t) - C\Delta x \frac{\partial V(x,t)}{\partial t} + -G\Delta x V(x,t) = I(x+\Delta x, t) \end{cases}$$

Move the term from the left most to the right

$$\begin{cases} -L\Delta x \frac{\partial I(x,t)}{\partial t} - R\Delta x I(x,t) = V(x+\Delta x, t) - V(x,t) \\ -C\Delta x \frac{\partial V(x,t)}{\partial t} - G\Delta x V(x,t) = I(x+\Delta x, t) - I(x,t) \end{cases}$$

Divide both side with Δx

$$\begin{cases} -L \frac{\partial I(x,t)}{\partial t} - RI(x,t) = \frac{V(x+\Delta x, t) - V(x,t)}{\Delta x} \\ -C \frac{\partial V(x,t)}{\partial t} - GV(x,t) = \frac{I(x+\Delta x, t) - I(x,t)}{\Delta x} \end{cases}$$

Take limit $\Delta x \to 0$

$$\begin{cases} -L \frac{\partial I(x,t)}{\partial t} - RI(x,t) = \frac{\partial V(x,t)}{\partial x} \\ -C \frac{\partial V(x,t)}{\partial t} - GV(x,t) = \frac{\partial I(x,t)}{\partial x} \end{cases}$$

Perform $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}$ on both equations

$$\begin{cases} -L\dfrac{\partial I}{\partial t} - RI = \dfrac{\partial V}{\partial x} & (1) \\ -C\dfrac{\partial V}{\partial t} - GV = \dfrac{\partial I}{\partial x} & (2) \\ -L\dfrac{\partial^2 I}{\partial t^2} - R\dfrac{\partial I}{\partial t} = \dfrac{\partial^2 V}{\partial t \partial x} & (3) \\ -C\dfrac{\partial^2 V}{\partial t^2} - G\dfrac{\partial V}{\partial t} = \dfrac{\partial^2 I}{\partial x \partial t} & (4) \\ -L\dfrac{\partial^2 I}{\partial t \partial x} - R\dfrac{\partial I}{\partial x} = \dfrac{\partial^2 V}{\partial x^2} & (5) \\ -C\dfrac{\partial^2 V}{\partial t \partial x} - G\dfrac{\partial V}{\partial x} = \dfrac{\partial^2 I}{\partial x^2} & (6) \end{cases}$$

Put (3) into(6) and then put it into (1)

$$-C\left(-L\frac{\partial^2 I}{\partial t^2} - R\frac{\partial I}{\partial t}\right) - G\frac{\partial V}{\partial x} = \frac{\partial^2 I}{\partial x^2} \qquad (3) \to (6)$$

$$-L\frac{\partial I}{\partial t} - RI = -\frac{C}{G}\left(-L\frac{\partial^2 I}{\partial t^2} - R\frac{\partial I}{\partial t}\right) - \frac{1}{G}\frac{\partial^2 I}{\partial x^2} \qquad ((3) \to (6)) \to (1)$$

Re-arrange

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + LG)\frac{\partial I}{\partial t} + RIG$$

If put (4) into (5) and then put it into (2)

$$-L\left(-C\frac{\partial^2 V}{\partial t^2} - G\frac{\partial V}{\partial t}\right) - R\frac{\partial I}{\partial x} = \frac{\partial^2 V}{\partial x^2} \qquad (4) \to (5)$$

$$-C\frac{\partial V}{\partial t} - GV = -\frac{L}{R}\left(-C\frac{\partial^2 V}{\partial t^2} - G\frac{\partial V}{\partial t}\right) - \frac{1}{R}\frac{\partial^2 V}{\partial x^2} \qquad ((4) \to (5)) \to (2)$$

Re-arrange

$$\frac{\partial^2 V}{\partial x^2} = CL\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + GVR$$

Therefore the 2 Telegrahper's Equations are

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} = CL\dfrac{\partial^2 V}{\partial t^2} + (LG + RC)\dfrac{\partial V}{\partial t} + GVR \\ \\ \dfrac{\partial^2 I}{\partial x^2} = LC\dfrac{\partial^2 I}{\partial t^2} + (RC + LG)\dfrac{\partial I}{\partial t} + RIG \end{cases}$$

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