Solving Telegrapher's Equation by Klein-Gordon Equation

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Reference : EqWorld

Transformation of Telegrapher's Equation into Stand 2nd order Hyperbolic Linear PDE

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} = CL\dfrac{\partial^2 V}{\partial t^2} + (RC+LG)\dfrac{\partial V}{\partial t} + GRV \\ \dfrac{\partial^2 I}{\partial x^2} = LC\dfrac{\partial^2 I}{\partial t^2} + (RC+LG)\dfrac{\partial I}{\partial t} + GRI \end{cases}$$

This is a Second-Order Hyperbolic Partial Differential Equations

The general form othe this equation is

$$\frac{\partial^2 F(x,t)}{\partial t^2} + k\frac{\partial F(x,t)}{\partial t} = a^2\frac{\partial^2 F(x,t)}{\partial x^2} + bF(x,t)$$

Therefore

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} + \dfrac{RC+LG}{LC}\dfrac{\partial V}{\partial t} = \dfrac{1}{CL}\dfrac{\partial^2 V}{\partial x^2} - \dfrac{GR}{LC}V \\ \dfrac{\partial^2 I}{\partial t^2} + \dfrac{RC+LG}{LC}\dfrac{\partial I}{\partial t} = \dfrac{1}{CL}\dfrac{\partial^2 I}{\partial x^2} - \dfrac{GR}{LC}I \end{cases}$$

Where

$$k = \frac{RC+LG}{LC} \qquad a = \frac{1}{\sqrt{LC}} \qquad b = -\frac{GR}{LC}$$

Thus since V , I are in the form of $F(x,t)$, we now solve $F(x,t)$

Solving the Stand 2nd order Hyperbolic Linear PDE by transformation into Klein-Gordon equation

By a **VERY GENIUS substitution** $F(x,t) = e^{-\frac{1}{2}kt}u(x,t)$

Put it into the equation

$$\frac{\partial^2 F(x,t)}{\partial t^2} + k\frac{\partial F(x,t)}{\partial t} = a^2\frac{\partial^2 F(x,t)}{\partial x^2} + bF(x,t)$$

$\iff$

$$\frac{\partial^2 e^{-\frac{1}{2}kt}u(x,t)}{\partial t^2} + k\frac{\partial e^{-\frac{1}{2}kt}u(x,t)}{\partial t} = a^2\frac{\partial^2 e^{-\frac{1}{2}kt}u(x,t)}{\partial x^2} + be^{-\frac{1}{2}kt}u(x,t)$$

Use short hand notation

$$(eu)_{tt} + k\,(eu)_t = a^2\,(eu)_{xx} + beu$$

Consider $(eu)_t$

$$(eu)_t = e_t u + eu_t = -\frac{k}{2}eu + eu_t$$

Thus for $(eu)_{tt}$

$$
\begin{aligned}
(eu)_{tt} &= ((eu)_t)_t \\
&= (e_t u + e u_t)_t \\
&= e_{tt} u + e_t u_t + e_t u_t + e u_{tt} \\
&= e_{tt} u + 2 e_t u_t + e u_{tt} \\
&= \frac{k^2}{4} eu - keu_t + eu_{tt}
\end{aligned}
$$

$(eu)_{xx}$ is simple, since e is indepdent of x

$$(eu)_{xx} = eu_{xx}$$

And therefore

$$(eu)_{tt} + k\,(eu)_t = a^2\,(eu)_{xx} + beu$$

$\Longleftrightarrow$

$$\frac{k^2}{4} eu - keu_t + eu_{tt} + -\frac{k^2}{2} eu + keu_t = a^2 eu_{xx} + beu$$

Grouping same terms and take e out

$$u_{tt} + -\frac{k^2}{4} u = a^2 u_{xx} + bu$$

Rearrange

$$u_{tt} = a^2 u_{xx} + \left(b + \frac{k^2}{4}\right) u$$

Or

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \left(b + \frac{k^2}{4}\right) u(x,t)$$

This is called **KleinGordon equation**

Solving KleinGordon equation

Let $-\beta = b + \dfrac{k^2}{4}$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \beta u(x,t)$$

The general solutions of Klein-Gordon equation are

$$
\begin{array}{cc}
w(x,t) = \left\{ \begin{array}{c} \cos\lambda x \\ \sin\lambda x \end{array} \right\} \lambda x \left[A\cos\mu t + B\sin\mu t\right] & b + a^2\lambda^2 = \mu^2 \\
w(x,t) = e^{\pm\mu t}\left[A\cos\mu x + B\sin\mu x\right] & b + a^2\lambda^2 = -\mu^2 \\
w(x,t) = e^{\pm\lambda x}\left[A\cos\mu t + B\sin\mu t\right] & b - a^2\lambda^2 = \mu^2 \\
w(x,t) = e^{\pm\lambda x}\left[Ae^{\mu t} + Be^{-\mu t}\right] & b - a^2\lambda^2 = -\mu^2 \\
w(x,t) = AJ_0(\xi) + BY_0(\xi) & \xi = \dfrac{\sqrt{b}}{a}\sqrt{a^2\,(t + C_1)^2 - (x + C_2)^2}, b > 0 \\
w(x,t) = AI_0(\xi) + BK_0(\xi) & \xi = \dfrac{\sqrt{-b}}{a}\sqrt{a^2\,(t + C_1)^2 - (x + C_2)^2}, b < 0
\end{array}
$$

(How to solve the Klein-Gordon Equation will be discussed in another document)

Thus , by solving the Klein-Gordon equation, the Telegrapher equation can be solved

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