Transmission Line Theory for Lumped System

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1 Telegrapher Equation from KVL & KCL

KVL :

$$v(z,t) - R\Delta z \cdot i\,(z,t) - L\Delta z \cdot \frac{\partial i\,(z,t)}{\partial t} - v\,(z+\Delta z,t) = 0$$

$$\frac{v\,(z+\Delta z,t) - v(z,t)}{\Delta z} = -R \cdot i\,(z,t) - L\frac{\partial i\,(z,t)}{\partial t}$$

KCL :

$$i(z,t) - G\Delta z \cdot v\,(z+\Delta z,t) - C\Delta z \cdot \frac{\partial v\,(z+\Delta z,t)}{\partial t} - i\,(z+\Delta z,t) = 0$$

$$\frac{i\,(z+\Delta z,t) - i\,(z,t)}{\Delta z} = -Gv\,(z,t) - C\frac{\partial v\,(z+\Delta z,t)}{\partial t}$$

Shrink segment $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{v\,(z+\Delta z,t) - v(z,t)}{\Delta z} = \frac{\partial v(z,t)}{\partial z} = -Ri\,(z,t) - L\frac{\partial i\,(z,t)}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{i\,(z+\Delta z,t) - i\,(z,t)}{\Delta z} = -Gv\,(z,t) - C\frac{\partial v\,(z,t)}{\partial t}$$

Using phasor

$$\frac{dV(z)}{dz} = -\left(R + j\omega L\right) I(z)$$

$$\frac{dI(z)}{dz} = -\left(G + j\omega C\right) V(z)$$

$$Telegrapher\ Equation$$

In lumped system, $\omega = 0$, L part is zero. For very short wire, $R = 0$, so in lumped system, Telegrapher Equation is not obvious in lumped system.

2 Wave Equation form Telegrapher Eqaution

$$\begin{cases} \dfrac{dV(z)}{dz} = -\left(R + j\omega L\right) I(z) \\ \\ \dfrac{dI(z)}{dz} = -\left(G + j\omega C\right) V(z) \end{cases} \xrightarrow{\frac{d}{dz}} \begin{cases} \dfrac{d^2V(z)}{dz^2} = -\left(R + j\omega L\right) \dfrac{dI(z)}{dz} \\ \\ \dfrac{d^2I(z)}{dz^2} = -\left(G + j\omega C\right) \dfrac{dV(z)}{dz} \end{cases}$$

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} = \left(R + j\omega L\right)\left(G + j\omega C\right) V(z) \\ \\ \dfrac{d^2I(z)}{dz^2} = \left(G + j\omega C\right)\left(R + j\omega L\right) I(z) \end{cases} \rightarrow \begin{cases} \dfrac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\ \\ \dfrac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \\ \gamma = \alpha + j\beta = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} \end{cases}$$

Recall,

$$\frac{d^2y(t)}{dt^2} + A\frac{dy(t)}{dt} + By(t) = 0$$

Let $y(t) = ke^{\lambda t}$

$$\lambda^2 ke^{\lambda t} + A\lambda ke^{\lambda t} + Bke^{\lambda t} = 0 \rightarrow \lambda^2 + A\lambda + B = 0$$

For $A = 0$, $\lambda = \pm j\sqrt{B}$

$$y(t) = ke^{+\sqrt{B}t} + ke^{-\sqrt{B}t}$$

So for

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\ \\ \dfrac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \\ \gamma = \alpha + j\beta = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} \end{cases} \qquad \begin{cases} V(z) = V_1 e^{-\gamma t} + V_2 e^{+\gamma t} \\ I(z) = I_1 e^{-\gamma t} + I_2 e^{+\gamma t} \end{cases}$$

$$\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = \left(R + j\omega L\right)\left(G + j\omega C\right) = \left(RG - \omega^2 LC\right) + j\omega\left(LG + RC\right)$$

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}}$$

$$\beta = \sqrt{\frac{-RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}}$$

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