Telegrapher's Equation for time-hamonic signals

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In the derivation of the telegrapher's equations, after the KCL and KVL is used , the following 2 equations are obtained

$$\begin{cases} -L\dfrac{\partial I(x,t)}{\partial t} - RI(x,t) = \dfrac{\partial V(x,t)}{\partial x} \\ -C\dfrac{\partial V(x,t)}{\partial t} - GV(x,t) = \dfrac{\partial I(x,t)}{\partial x} \end{cases}$$

For time harmonic signals, phasor can be used

$$V(x,t) = Re\left(V(x)e^{j\omega t}\right)$$

In phasor,

$$\frac{\partial}{\partial t} = j\omega$$

Thus

$$\begin{cases} -\left(R + j\omega L\right) I(x) = \dfrac{\partial V(x)}{\partial x} \\ -\left(G + j\omega C\right) V(x) = \dfrac{\partial I(x)}{\partial x} \end{cases}$$

Rearrange

$$\begin{cases} \dfrac{\partial^2 V(x)}{\partial x} = \left(G + j\omega C\right)\left(R + j\omega L\right) V(x) \\ \dfrac{\partial^2 I(x)}{\partial x} = \left(G + j\omega C\right)\left(R + j\omega L\right) I(x) \end{cases}$$

Rearrange

$$\begin{cases} \dfrac{\partial^2 V(x)}{\partial x} - \gamma^2 V(x) = 0 \\ \dfrac{\partial^2 I(x)}{\partial x} - \gamma^2 I(x) = 0 \end{cases} \qquad \gamma = \sqrt{\left(G + j\omega C\right)\left(R + j\omega L\right)}$$

The comples propagation constant can be expressed as

$$\gamma = \alpha + j\beta \qquad \begin{aligned} \alpha &= \sqrt{\frac{RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}} \\ \beta &= \sqrt{\frac{-RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}} \end{aligned}$$

(For derivation of α and β , please refer to the document on propagation constant)

Then how to solve for V and I ?

$$\left(\frac{\partial^2}{\partial x} - \gamma^2\right)\left\{ \begin{array}{c} V \\ I \end{array} \right\} = 0$$

The V and I can be solved using traditional method : Let

$$V = Ae^{\lambda x}$$

Put it into the ODE

$$A\lambda^2 e^{\lambda x} - \gamma^2 Ae^{\lambda x} = 0$$

$\Longleftrightarrow$

$$\lambda^2 = \gamma^2$$

Or

$$\lambda = \pm\gamma$$

Therefore , the solution is

$$V = V_0^- e^{\gamma x} \quad \text{or} \quad V = V_0^+ e^{-\gamma x}$$

The solution should be the super-position of the 2 solution, thus

$$\begin{cases} V(x) = V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x} \\ I(x) = I_0^+ e^{-\gamma x} + I_0^- e^{\gamma x} \end{cases}$$

Then the unknowns are V_0^+ , V_0^- , I_0^+ , I_0^- , 2 equations can not solve for 4 unknowns

Thus we need the help of Ohm's Law , consider the euqation

$$-(R + j\omega L)\, I(x) = \frac{\partial V(x)}{\partial x}$$

Put in the solution of the V

$$-(R + j\omega L)\, I(x) = \frac{\partial}{\partial x}\left(V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}\right)$$

$\Longleftrightarrow$

$$-(R + j\omega L)\, I(x) = -\gamma V_0^+ e^{-\gamma x} + \gamma V_0^- e^{\gamma x}$$

$\Longleftrightarrow$

$$(R + j\omega L)\, I(x) = \gamma\left(V_0^+ e^{-\gamma x} - V_0^- e^{\gamma x}\right)$$

By $V = IR$

$$I(x) = \underbrace{\frac{\gamma}{R + j\omega L}}_{Z^{-1}}\left(V_0^+ e^{-\gamma x} - V_0^- e^{\gamma x}\right)$$

Thus

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(G + j\omega C)(R + j\omega L)}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

And therefore, the solutions are now

$$V(x) = V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}$$

$$I(x) = \frac{V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}}{Z_0}$$

Where V_0^+ , V_0^- can be solved by boundary conditions

To get the time domain form, perform the inverse phasor , recall that a complex number can be expressed as magnitude and phase

$$V_0^+ = |V_0^+| \angle V_0^+ \qquad V_0^- = |V_0^-| \angle V_0^-$$

Thus

$$V(x,t) = \mathrm{Re}\left(V(z)e^{j\omega t}\right)$$

$$= \mathrm{Re}\left(\left(V_0^+ e^{-\gamma x} + V_0^- e^{\gamma x}\right) e^{j\omega t}\right)$$

$$= \mathrm{Re}\left(\left(|V_0^+| \angle V_0^+ e^{-\alpha x} e^{-j\beta x} + |V_0^-| \angle V_0^- V_0^+ e^{\alpha x} e^{j\beta x}\right) e^{j\omega t}\right)$$

$$= |V_0^+| e^{-\alpha x} \cos\left(\omega t - \beta x + \angle V_0^+\right) + |V_0^-| e^{\alpha x} \cos\left(\omega t + \beta x + \angle V_0^+\right)$$

To obtain the phase velocity , consider

$$\omega t - \beta x + \angle V_0^+ = \text{constant}$$

Then

$$x = \frac{\omega t}{\beta} + \frac{\angle V_0^+}{\beta} - \frac{\text{constant}}{\beta}$$

Where

$$v_{phase} = \frac{\partial x}{\partial t} = \frac{\omega}{\beta}$$

Since

$$v_{phase} = f\lambda$$

$\iff$

$$\lambda = \frac{v_{phase}}{f} = \frac{\omega/\beta}{f} = \frac{2\pi}{\beta}$$

Lossless Transmission Line

Lossless : $R = G = 0$

$$\gamma = j\omega\sqrt{LC} = \alpha + j\beta$$

$$\alpha = 0 \;\; \beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$v_p = \frac{1}{\sqrt{LC}} \qquad \lambda = \frac{1}{f\sqrt{LC}}$$

$-END-$