Parameters in lossless transmission line with a load

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The Reflection Coefficient

The general voltage and current are

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}}{Z_0}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

Set the coordinate at $z = 0$ (at the load position , which is the boundary condition), we can find load voltage and load current

$$V_L = V(0) = V_0^+ + V_0^-$$

$$I_L = I(0) = \frac{V_0^+ - V_0^-}{Z_0}$$

And thus the load impedance can be found using Ohm's Law

$$Z_L = \frac{V_L}{I_L} = \frac{V_0^+ + V_0^-}{\frac{V_0^+ - V_0^-}{Z_0}} = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}$$

With this equation we can find and expression for $\frac{V^-}{V^+}$

$$Z_L = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}$$

$$\Longleftrightarrow \quad \frac{Z_L}{Z_0} = \frac{1 + \frac{V_0^-}{V_0^+}}{1 - \frac{V_0^-}{V_0^+}}$$

$$\Longleftrightarrow \quad \frac{V_0^-}{V_0^+} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1}$$

$$\Longleftrightarrow \quad \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This ratio, is called $Reflection\ Coefficient$, often expressed as Γ or R

$$\frac{V_0^-}{V_0^+} = \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

The Γ Circle

With the help of Γ , we can rewrite all the V and I

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ e^{-j\beta z}\left[1 + \frac{V_0^-}{V_0^+} e^{j2\beta z}\right] = V_0^+ e^{-j\beta z}\left[1 + \Gamma e^{j2\beta z}\right]$$

$$I(z) = \frac{V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}}{Z_0} = \frac{V_0^+}{Z_0} e^{-j\beta z}\left[1 - \frac{V_0^-}{V_0^+} e^{j2\beta z}\right] = \frac{V_0^+}{Z_0} e^{-j\beta z}\left[1 - \Gamma e^{j2\beta z}\right]$$

If only magnitude is considered , by $|e^{j\theta}| = 1$

$$|V(z)| = |V_0^+||1 + \Gamma e^{j2\beta z}|$$

$$|I(z)| = \frac{|V_0^+|}{|Z_0|}|1 - \Gamma e^{j2\beta z}|$$

What are these eqautions ? They are circle equations !

Review of Complex Algebra about circle
Consider

$$r = |a + z| \quad a \in \mathbb{R}, z \in \mathbb{C}$$

Since $z = x + jy$, thus

$$r = |a + z| = |(a + x) + jy| = \sqrt{(a + x)^2 + y^2}$$

i.e.

$$(a + x)^2 + y^2 = r^2 \qquad \text{circle with center (a,0) , radius r}$$

Standing Wave Ratio

Consider a ratio , the V_{max} to V_{min} , and call this ratio as $standing\,wave\,ratio$

$$SWR = \frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Time Average Power

Consider the power

$$P_{Avg} = \frac{1}{T}\int_t^{t+T} P(z,t)dt = \frac{1}{T}\int_t^{t+T} I(z,t)V(z,t)dt$$

Recall that time average power can be found by using complex phasor

$$P_{Avg} = \frac{1}{2}\mathbf{Re}\left\{VI*\right\}$$

$$V(z) = V_0^+ e^{-j\beta z}\left[1 + \Gamma e^{j2\beta z}\right]$$

$$I(z)* = \left(\frac{V_0^+}{Z_0} e^{-j\beta z}\left[1 - \Gamma e^{j2\beta z}\right]\right)* = \frac{V_0^+}{Z_0} e^{j\beta z}\left[1 - \Gamma e^{-j2\beta z}\right]$$

$$= \frac{1}{2}\mathbf{Re}\left\{\left[V_0^+ e^{-j\beta z}\left[1 + \Gamma e^{j2\beta z}\right]\right]\left[\frac{V_0^+}{Z_0} e^{j\beta z}\left[1 - \Gamma e^{-j2\beta z}\right]\right]\right\}$$

$$= \frac{|V_0^+|^2}{2Z_0}\left[1-|\Gamma|^2\right]$$

$$= \frac{|V_0^+|^2}{2Z_0} - |\Gamma|^2\frac{|V_0^+|^2}{2Z_0}$$

$$= P^I + P^R$$

That is

$$P_{Avg} = \frac{|V_0^+|^2}{2Z_0}\left[1-|\Gamma|^2\right] = P^I + P^R$$

$$P^I = \frac{|V_0^+|^2}{2Z_0} \qquad \text{Incident power}$$

$$P^R = |\Gamma|^2\frac{|V_0^+|^2}{2Z_0} \qquad \text{Reflected power}$$

Return Loss

Consider the ratio of incident power and reflected power

$$\frac{P^I}{P^R} = \frac{\frac{|V_0^+|^2}{2Z_0}}{|\Gamma|^2\frac{|V_0^+|^2}{2Z_0}} = \frac{1}{|\Gamma|^2}$$

Turn it into dB

$$10\log\frac{P^I}{P^R} = 10\log\frac{1}{|\Gamma|^2} = -10\log|\Gamma|^2 = -20\log|\Gamma|$$

Such parameter is called Return Loss

$$ReturnLoss = -20\log|\Gamma| = -20\log\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|$$

Impdeance of lossless transmission line

The impedance in a location z is

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-j\beta z}\left(1+\Gamma e^{j2\beta z}\right)}{\frac{V_0^+}{Z_0}e^{-j\beta z}\left(1-\Gamma e^{j2\beta z}\right)} = Z_0\frac{1+\Gamma e^{j2\beta z}}{1-\Gamma e^{j2\beta z}}$$

Express Γ as Z_0 , Z_L

$$Z(z) = Z_0\frac{1+\frac{Z_L-Z_0}{Z_L+Z_0}e^{j2\beta z}}{1-\frac{Z_L-Z_0}{Z_L+Z_0}e^{j2\beta z}} = Z_0\frac{Z_L+Z_0+(Z_L-Z_0)\,e^{j2\beta z}}{Z_L+Z_0-(Z_L-Z_0)\,e^{j2\beta z}} = Z_0\frac{Z_L(1+e^{j2\beta z})+Z_0(1-e^{j2\beta z})}{Z_L(1-e^{j2\beta z})-Z_0(1+e^{j2\beta z})}$$

Multiply $\frac{e^{-j\beta z}}{e^{-j\beta z}}$

$$Z(z) = Z_0\frac{Z_L(e^{-j\beta z}+e^{j\beta z})+Z_0(e^{-j\beta z}-e^{j\beta z})}{Z_L(e^{-j\beta z}-e^{j\beta z})-Z_0(e^{-j\beta z}+e^{j\beta z})}$$

By Euler's Formula

$$Z(z) = Z_0\frac{2Z_L\cos\beta z - 2jZ_0\sin\beta z}{2jZ_L\sin\beta z - 2Z_0\cos\beta z} = Z_0\frac{Z_L - jZ_0\tan\beta z}{Z_0 - jZ_L\tan\beta z}$$

Thus

$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan \beta z}{Z_0 - jZ_L \tan \beta z}$$

For example at position $z = -l$

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

Different case of $Z(z)$

$$\frac{V_0^-}{V_0^+} = \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$V(z) = V_0^+ e^{-j\beta z} \left[1 + \Gamma e^{j2\beta z}\right] \qquad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} \left[1 - \Gamma e^{j2\beta z}\right]$$

$$SWR = \frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$P_{Avg} = \frac{|V_0^+|^2}{2Z_0} \left[1 - |\Gamma|^2\right] = P^I + P^R$$

$$ReturnLoss = -20 \log |\Gamma|$$

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

When $Z_L = Z_0$ (matched)

$$\Gamma = 0$$

$$V(z) = V_0^+ e^{-j\beta z} \qquad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}$$

$$SWR = 1$$

$$P_{Avg} = P^I$$

$$ReturnLoss = \infty dB$$

$$Z(z) = Z_0 \text{ (length indepdent !)}$$

There is no reflection

Since $Z = Z_0 \in \mathbb{R}$, thus the transmission line act as a resistive element

When $Z_L = 0 \neq Z_0$ (not match)

$$\Gamma = -1$$

$$V(z) = V_0^+ \left[e^{-j\beta z} - e^{j\beta z}\right] = -jV_0^+ \sin \beta z \qquad I(z) = 2\frac{V_0^+}{Z_0} \cos \beta z$$

$$SWR = \infty$$

$$P_{Avg} = 0$$

$$ReturnLoss = 0dB$$

$$Z(-l) = jZ_0 \tan \beta l$$

All reflected back

Since $Z = jZ_0 \tan \beta l$ $\in$pure imaginary , the transmission line act as a reactive element , inductor / capactor depends on l for the sign of tan

When $Z_L = \infty \neq Z_0$ (not match)

$$\Gamma = 1$$

$$V(z) = 2V_0^+ \cos \beta z \qquad I(z) = -2j \frac{V_0^+}{Z_0} \sin \beta z$$

$$SWR = \frac{|V(z)|_{max}}{|V(z)|_{min}} = \infty$$

$$P_{Avg} = 0$$

$$ReturnLoss = \infty$$

$$Z(z) = -jZ_0 \cot \beta z$$

Since $Z = -jZ_0 \cot \beta l$ $\in$pure imaginary , the transmission line act as a reactive element , inductor / capactor depends on l for the sign of cot

Transmission Coefficient

Consider 2 transmission lines connected together with the second one have no reflection (matched , so $Z_{02} = Z_L$)

$$V(z) = V_0^+ \left(e^{-j\beta} + \Gamma e^{j\beta z} \right) \qquad z < 0$$

$$V(z) = V_0^+ T e^{-j\beta z} \qquad z > 0$$

By boundary condition

$$V_{Z>0}(z=0) = V_{Z>0}(z=0)$$

$\iff$

$$T = 1 + \Gamma$$

Where T is the transmission coefficient

$$T = \frac{2Z_L}{Z_L + Z_0}$$

Remember that $Z_L = Z_{02}$ as the second transmission line is matched

Insertion Loss

For matched case

$$P_{av} = P^I = \frac{|V_0^+|^2}{2Z_0}$$

When transmission line is connected

$$P_{av} = P^T = |T|^2 \frac{|V_0^+|^2}{2Z_0}$$

Thus the insertion loss

$$InsertionLoss = 10 \log \frac{P^I}{P^T} = -20 \log |T| \ dB$$

$$-END-$$