Lossy and Low Loss Transmission Line

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1 Lossy Case

General Lossy Transmission Line

$\gamma = \alpha + j\beta$, $\alpha \neq 0$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$V = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I = \frac{V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}}{Z_0}$$

Apply the reflection coefficient Γ

$$V = V_0^+ e^{-\gamma z} \left(1 + \Gamma e^{2\gamma z}\right)$$

$$I = \frac{V_0^+ e^{-\gamma z} \left(1 - \Gamma e^{2\gamma z}\right)}{Z_0}$$

$$\Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^-}{V_0^+} e^{2\gamma z} = |\Gamma| e^{2\alpha z} e^{2\beta z}$$

Thus the input impedance is thus

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-\gamma z} \left(1 + \Gamma e^{2\gamma z}\right)}{\frac{V_0^+ e^{-\gamma z} \left(1 - \Gamma e^{2\gamma z}\right)}{Z_0}} = Z_0 \frac{1 + \Gamma e^{2\gamma z}}{1 - \Gamma e^{2\gamma z}}$$

$$= Z_0 \frac{e^{-\gamma z} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{\gamma z}}{e^{-\gamma z} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{\gamma z}}$$

$$= Z_0 \frac{(Z_L + Z_0) e^{-\gamma z} + (Z_L - Z_0) e^{\gamma z}}{(Z_L + Z_0) e^{-\gamma z} - (Z_L - Z_0) e^{\gamma z}}$$

$$= Z_0 \frac{Z_L \left(e^{-\gamma z} + e^{\gamma z}\right) + Z_0 \left(e^{-\gamma z} - e^{\gamma z}\right)}{Z_L \left(e^{-\gamma z} - e^{\gamma z}\right) + Z_0 \left(e^{-\gamma z} + e^{\gamma z}\right)}$$

$$= Z_0 \frac{2 Z_L \cosh \gamma z - 2 Z_0 \sinh \gamma z}{-2 Z_L \sinh \gamma z + 2 Z_0 \cosh \gamma z}$$

$$= Z_0 \frac{Z_L \cosh \gamma z - Z_0 \sinh \gamma z}{Z_0 \cosh \gamma z - Z_L \sinh \gamma z}$$

$$= Z_0 \frac{Z_L - Z_0 \tanh \gamma z}{Z_0 - Z_L \tanh \gamma z}$$

Consider $z = -l$

$$Z_{In}(-l) = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}$$

2 Low Loss case

$$R \ll \omega L \qquad G \ll \omega C$$

Derivation of the γ
To apply the low loss condition on γ

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{j\omega C \left(\frac{R}{j\omega L} + 1\right) j\omega C \left(\frac{G}{j\omega C} + 1\right)}$$

$$= j\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\sqrt{1 + \frac{R}{j\omega L} + \frac{G}{j\omega C} - \frac{RG}{\omega^2 LC}}$$

Since $R \ll \omega L \qquad G \ll \omega C$

$$\approx j\omega\sqrt{LC}\sqrt{1 + \left(\frac{R}{j\omega L} + \frac{G}{j\omega C}\right)}$$

Recall, Taylor's Expansion

$$f(x) = f(0) + \frac{f^{(1)}(0)x}{1!} + \frac{f^{(2)}(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + ...$$

Thus for $f = \sqrt{1+x}$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + ...$$

For $x \ll 1$

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

Thus

$$\gamma \approx j\omega\sqrt{LC}\sqrt{1 + \left(\frac{R}{j\omega L} + \frac{G}{j\omega C}\right)}$$

$$\approx j\omega\sqrt{LC}\left(1 + \frac{1}{2}\left(\frac{R}{j\omega L} + \frac{G}{j\omega C}\right)\right)$$

$$= j\omega\sqrt{LC}\left(1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right)$$

$$= j\omega\sqrt{LC} + \frac{\omega\sqrt{LC}}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)$$

$$= j\omega\sqrt{LC} + \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

Thus, by $\gamma = \alpha + j\beta$

$$\alpha \approx \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

$$\beta \approx \omega\sqrt{LC}$$

The V , I expression

$$V = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad I = \frac{V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}}{Z_0}$$

Using Γ

$$V(z) = V_0^+ e^{-\gamma z}\left(1 + \Gamma e^{2\gamma z}\right) \qquad I(z) = \frac{V_0^+ e^{-\gamma z}\left(1 - \Gamma e^{2\gamma z}\right)}{Z_0}$$

For $z = -l$

$$V = V_0^+ e^{\gamma l}\left(1 + \Gamma e^{-2\gamma l}\right) \qquad I = \frac{V_0^+ e^{\gamma l}\left(1 - \Gamma e^{-2\gamma l}\right)}{Z_0}$$

The power

$$P_{av} = \frac{1}{2}\mathrm{Re}\left\{VI*\right\}$$

$$= \frac{1}{2}\mathbf{Re}\left\{\left(V_0^+ e^{\gamma l}\left(1 + \Gamma e^{-2\gamma l}\right)\right)\left(\frac{V_0^+ e^{\gamma l}\left(1 - \Gamma e^{-2\gamma l}\right)}{Z_0}\right)\right\}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\mathbf{Re}\left\{e^{2\gamma l}\left(1 + \Gamma e^{-2\gamma l}\right)\left(1 - \Gamma e^{-2\gamma l}\right)\right\}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\mathbf{Re}\left\{e^{2\gamma l}\left(1 - |\Gamma|^2 e^{-4\gamma l}\right)\right\}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\mathbf{Re}\left\{e^{2\gamma l} - |\Gamma|^2 e^{-2\gamma l}\right\}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\left\{e^{2\alpha l} - |\Gamma|^2 e^{-2\alpha l}\right\}$$

Therefore

$$P_{loss} = P_{IN} - P_{Load}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\left\{e^{2\alpha l} - |\Gamma|^2 e^{-2\alpha l}\right\} - \frac{\left|V_0^+\right|^2}{2Z_0}\left\{1 - |\Gamma|^2\right\}$$

$$= \frac{\left|V_0^+\right|^2}{2Z_0}\left\{e^{2\alpha l} - 1 + |\Gamma|^2\left(1 - e^{-2\alpha l}\right)\right\}$$

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